Series

Summation of finite series using the method of differences

You may remember from C1 (or C2??) how to derive the sum to n terms of an arithmetic series with a first term n and common difference d:

$$\begin{array}{lllllll} & 1 & 2 & 3 & 4 & \dots & n \\ S_n = & a & + (a+d) & + (a+2d) & + (a+3d) & + \dots & + [a+(n-1)d] \end{array}$$

reversing $\quad S_n = a+(n-1)d + a+(n-2)d + \dots + a$

adding $\quad 2S_n = 2a+(n-1)d \quad 2a+(n-1)d + \dots + 2a+(n-1)d$

i.e. $\quad 2S_n = n$ lots of $2a+(n-1)d$

$\therefore \quad 2S_n = n[2a+(n-1)d]$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

The same method can be employed to sum the following series:

$$1 + 2 + 3 + 4 + \dots + n$$

This is written symbolically as $\sum_{r=1}^{n} r$

Now $\quad \sum_{r=1}^{n} r = 1 + 2 + 3 + 4 + \dots + (n-1) + n$

reversing $\quad \sum_{r=1}^{n} r = n + (n-1) + (n-2) + \dots + 1$

$n-1+2 \quad n-2+3$

adding. $\quad 2\sum_{r=1}^{n} r = (n+1) + (n+1) + (n+1) + \dots (n+1)$

$\therefore \quad 2\sum_{r=1}^{n} r = n(n+1) \quad \Rightarrow \quad \sum_{r=1}^{n} r = \frac{n}{2}(n+1).$

An alternative method would be to consider the identity

$$2r \equiv r(r+1) - (r-1)r$$

eg1 $\quad$ Use the identity $2r \equiv r(r+1) - (r-1)r$ to find $\sum_{r=1}^{n} r$

This method of summing a series is known as **the differences method.** The cancelling of nearly all the terms, is similar to the way in which the interior sections of a collapsible telescope disappear when it is compressed, so this effect is described as *telescoping a sum.*

At this level we are usually given a hint about what function to consider, but they tend to take the form:

If $\quad u_r \equiv f(r+1) - f(r)$

then $\quad \sum_{r=1}^{n} u_r = f(n+1) - f(1)$

eg2 Given that $f(r) = \dfrac{1}{r^2}$, show that

$$f(r) - f(r+1) = \frac{2r+1}{r^2(r+1)^2}$$

and hence find

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$$

eg3 By considering f(r) – f(r+1), where $f(r) = \dfrac{r+2}{r(r+1)}$

Prove that

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \frac{3}{2} - \frac{n+3}{(n+1)(n+2)}$$

Exercise 2A Pg 16

Eg 1 $\quad 2r \equiv r(r+1) - (r-1)r$

$r=1 \quad 2(1) = 1\cancel{(2)} \boxed{- (0)1}$

$r=2 \quad 2(2) = 2\cancel{(3)} - \cancel{1(2)}$

$r=3 \quad 2(3) = 3\cancel{(4)} - \cancel{2(3)}$

$r=4 \quad 2(4) = 4(5) - \cancel{3(4)}$

⋮

$r=n-1 \quad 2(n-1) \equiv \cancel{(n-1)}n - \cancel{(n-2)(n-1)}$

$r=n \quad 2(n) \equiv \boxed{n(n+1)} - \cancel{(n-1)n}$

Not cancelled

Adding $\therefore \quad 2\sum_{r=1}^{n} r = n(n+1) - 0$

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

eg 2 $f(r) = \frac{1}{r^2}$

Review Ex1 Q28

$$f(r) - f(r+1) = \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$$\frac{2r+1}{r^2(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2}$$

$$= \frac{r^2+2r+1-r^2}{\cancel{r^2(r^2+2r+1)}\ r^2(r+1)^2}$$

$$= \frac{2r+1}{r^2(r+1)^2}$$

$$\therefore \frac{2r+1}{r^2(r+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$$

$r=1$ $\quad \frac{2(1)+1}{(1)^2(1+1)^2} = \frac{1}{1^2} - \frac{1}{2^2}$

$r=2$ $\quad \frac{2(2)+1}{(2)^2(2+1)^2} = \frac{1}{2^2} - \frac{1}{3^2}$

$r=3$ $\quad \frac{2(3)+1}{(3)^2(3+1)^2} = \frac{1}{3^2} - \frac{1}{4^2}$

$r=4$ $\quad \frac{2(4)+1}{(4)^2(4+1)^2} = \frac{1}{4^2} - \frac{1}{5^2}$

$\vdots \quad \vdots \quad = \quad \vdots \quad \vdots$

$r=n-1$ $\quad \frac{2(n-1)+1}{(n-1)^2(n)^2} = \frac{1}{(n-1)^2} - \cancel{\frac{1}{n^2}}$

$r=n$ $\quad \frac{2(n)+1}{n^2(n+1)^2} = \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$

$\frac{1}{1^2} \quad \boxed{-\frac{1}{2^2} \quad \frac{1}{2^2}} \quad \boxed{-\frac{1}{3^2} \quad \frac{1}{3^2}} \quad \boxed{-\frac{1}{4^2} \quad \frac{1}{4^2}} \quad -\frac{1}{5^2} \ \ldots$

Adding $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - 1}{(n+1)^2} = \frac{n^2+2n+1-1}{(n+1)^2}$

$$= \frac{n(n+2)}{(n+1)^2}$$

Ex2A

① (a) $r = \frac{1}{2}\left(r(r+1) - r(r-1)\right)$

$= \frac{1}{2}\left(r^2 + r - r^2 + r\right)$

$= \frac{1}{2} \times 2r$

$= r$

(b) $r = \frac{1}{2}\left(r(r+1) - r(r-1)\right)$

$r=1 \quad 1 = \frac{1}{2}\cancel{(2)} - \boxed{\frac{1}{2}(0)}$

$r=2 \quad 2 = \frac{1}{2}\cancel{(6)} - \frac{1}{2}\cancel{(2)}$

$r=3 \quad 3 = \frac{1}{2}\cancel{(12)} - \frac{1}{2}\cancel{(6)}$

$r=4 \quad 4 = \frac{1}{2}\cancel{(20)} - \frac{1}{2}\cancel{(12)}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$r=n-1 \quad (n-1) = \cancel{\frac{1}{2}n(n-1)} - \frac{1}{2}\cancel{(n-1)(n-2)}$

$r=n \quad n = \boxed{\frac{1}{2}n(n+1)} - \frac{1}{2}\cancel{n(n-1)}$

Adding

$\sum n = \frac{1}{2}n(n+1)$

(2) $$\frac{1}{r(r+1)(r+2)} \equiv \frac{1}{2r(r+1)} - \frac{1}{2(r+1)(r+2)}$$

$r=1$ $$\frac{1}{1(2)(3)} \equiv \boxed{\frac{1}{2(2)}} - \cancel{\frac{1}{2(6)}}$$

$r=2$ $$\frac{1}{2(3)(4)} \equiv \cancel{\frac{1}{2(6)}} - \cancel{\frac{1}{2(12)}}$$

$r=3$ $$\frac{1}{3(4)(5)} \equiv \cancel{\frac{1}{2(12)}} - \cancel{\frac{1}{2(20)}}$$

$r=4$ $$\frac{1}{4(5)(6)} \equiv \cancel{\frac{1}{2(20)}} - \cancel{\frac{1}{2(30)}}$$

$\vdots$

$r=n-1$ $$\frac{1}{(n-1)n(n+1)} \equiv \cancel{\frac{1}{2n(n-1)}} - \cancel{\frac{1}{2n(n+1)}}$$

$r=n$ $$\frac{1}{n(n+1)(n+2)} \equiv \cancel{\frac{1}{2n(n+1)}} \boxed{- \frac{1}{2(n+1)(n+2)}}$$

Adding. $$\sum \frac{1}{n(n+1)(n+2)} \equiv \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$$\equiv \frac{(n+1)(n+2) - 2}{4(n+1)(n+2)}$$

$$\equiv \frac{n^2+3n+2-2}{4(n+1)(n+2)}$$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

③ (a) $$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$

$$= \frac{A(r+2)+rB}{r(r+2)}$$

$A+B=0$

$2A=1$

$A=\frac{1}{2} \quad B=-\frac{1}{2}$

$$\frac{1}{r(r+2)} \equiv \frac{1}{2r} - \frac{1}{2(r+2)}$$ ✱ Book is different (& wrong!)

(b)

$r=1 \quad \frac{1}{1(3)} \equiv \boxed{\frac{1}{2(1)}} - \cancel{\frac{1}{2(3)}}$

$r=2 \quad \frac{1}{2(4)} \equiv \boxed{\frac{1}{2(2)}} - \cancel{\frac{1}{2(4)}}$

$r=3 \quad \frac{1}{3(5)} \equiv \cancel{\frac{1}{2(3)}} - \frac{1}{2(5)}$

$r=4 \quad \frac{1}{4(6)} \equiv \cancel{\frac{1}{2(4)}} - \frac{1}{2(6)}$

$r=n-2 \quad \frac{1}{(n-2)(n)} \equiv \frac{1}{2(n-2)} - \cancel{\frac{1}{2n}}$

$r=n-1 \quad \frac{1}{(n-1)(n+1)} \equiv \cancel{\frac{1}{2(n-1)}} \boxed{- \frac{1}{2(n+1)}}$

$r=n \quad \frac{1}{n(n+2)} \equiv \cancel{\frac{1}{2n}} \boxed{- \frac{1}{2(n+2)}}$

$$\cancel{\sum \frac{1}{n(n+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+2)}}$$

$$\cancel{= \frac{3}{4} - \frac{1}{2(n+2)}}$$

$$\cancel{= \frac{3(n+2)-2}{4(n+2)} = \frac{3n+4}{4(n+2)}}$$

3b).

$$\sum \frac{1}{n(n+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{3n^2+9n+6-2n-4-2n-2}{4(n+1)(n+2)}$$

$$= \frac{3n^2+5n}{4(n+1)(n+2)}$$

$$= \frac{n(3n+5)}{4(n+1)(n+2)}$$

4) a)

$$\frac{1}{(r+2)(r+3)} = \frac{A}{r+2} + \frac{B}{r+3}$$

$$= \frac{1}{(r+2)} - \frac{1}{(r+3)}$$

$r=1$ $\quad \frac{1}{(3)(4)} \equiv \boxed{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$r=2$ $\quad \frac{1}{(4)(5)} = \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$r=3$ $\quad \frac{1}{(5)(6)} = \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$r=n-1$ $\quad \frac{1}{(n+1)(n+2)} \equiv \frac{1}{n+1} - \cancel{\frac{1}{n+2}}$

$r=n$ $\quad \frac{1}{(n+2)(n+3)} \equiv \cancel{\frac{1}{n+2}} \boxed{-\frac{1}{n+3}}$

Adding

$$\sum \frac{1}{(n+2)(n+3)} \equiv \frac{1}{3} - \frac{1}{n+3} = \frac{n+3-3}{3(n+3)} = \frac{n}{3(n+3)}.$$

(5). $4r^3 \equiv r^2(r+1)^2 - (r-1)^2 r^2$

$r=1$ $\quad 4(1)^3 \equiv 1^2\cancel{(2)^2} \boxed{- 0^2(1)^2}$

$r=2$ $\quad 4(2)^3 \equiv \cancel{(2)^2}(3)^2 - (1)^2\cancel{(2)^2}$

$r=3$ $\quad 4(3)^3 \equiv (3)^2\cancel{(4)^2} - \cancel{(2)^2}(3)^2$

$r=4$ $\quad 4(4)^3 \equiv (4)^2(5)^2 - (3)^2\cancel{(4)^2}$

$r=n-1$ $\quad 4(n-1)^3 \equiv \cancel{(n-1)^2}(n)^2 - \cancel{(n-2)^2}(n-1)^2$

$r=n$ $\quad 4n^3 \equiv \boxed{n^2(n+1)^2} - \cancel{(n-1)^2}n^2$

Adding. $\quad 4\sum n^3 \equiv n^2(n+1)^2$

$\sum n^3 \equiv \frac{n^2}{4}(n+1)^2$

6. $\frac{r}{(r+1)!} \equiv \frac{1}{r!} - \frac{1}{(r+1)!}$

$r=1 \quad \frac{1}{2!} \equiv \boxed{\frac{1}{1!}} - \cancel{\frac{1}{2!}}$

$r=2 \quad \frac{1}{3!} = \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}}$

$r=3 \quad \frac{1}{4!} = \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}}$

$\vdots$

$r=n-1 \quad \frac{1}{n!} = \cancel{\frac{1}{(n-1)!}} - \cancel{\frac{1}{n!}}$

$r=n \quad \frac{1}{(n+1)!} = \cancel{\frac{1}{n!}} \boxed{- \frac{1}{(n+1)!}}$

Adding

$$\sum \frac{1}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

7. $\frac{2r+1}{r^2(n+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$

$r=1 \quad \frac{2(1)+1}{(1)^2(2)^2} \equiv \boxed{\frac{1}{1^2}} - \cancel{\frac{1}{2^2}}$

$r=2 \quad \frac{2(2)+1}{(2^2)(3)^2} \equiv \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$

$r=3 \quad \frac{2(3)+1}{3^2\ 4^2} \equiv \cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$

$\vdots$

$r=(n-1) \quad \frac{2(n-1)+1}{(n-1)^2\ n^2} \equiv \cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}$

$r=n \quad \frac{2n+1}{n^2(n+1)^2} \equiv \cancel{\frac{1}{n^2}} \boxed{- \frac{1}{(n+1)^2}}$

7 contd Adding $\sum \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}$

$$= \frac{(n+1)^2 - 1}{(n+1)^2}$$

$$= \frac{n(n+2)}{(n+1)^2}$$