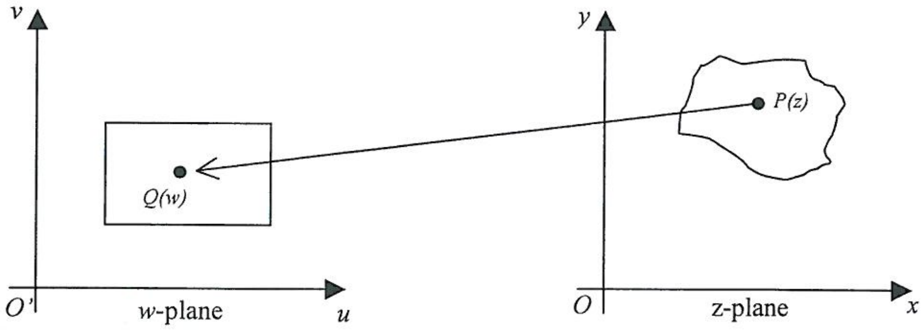
Transformations of Complex Variables

It is sometimes useful in more complicated applications of complex variable theory to associate with a complex number $z = x + iy$ another complex number $w = u + iv$ defined by a relation $w = f(z)$, where f is a known function. In this way a given point in the Argand diagram of z (the z-plane) usually becomes a different point in the Argand diagram of w (the w-plane), and, more importantly, a given region in the z-plane usually transforms into a different shaped region in the w-plane.

When dealing, therefore, with the variation of some quantity over a complicated region of the z-plane it is often convenient to make a transformation to the w-pane such that the region concerned becomes a simpler one to deal with (for instance, a circle or rectangle). This is shown diagrammatically below with the point *P(z)* being transformed into the point *Q(w)*.



If in fact to every point in the z-plane there corresponds just one point in the w-plane, and to every point in the w-plane there corresponds just one point in the z-plane, the transformation is one-to-one.

Eg ~~21~~ 18   Find the transforms of the circles $|z| = 1$ and $|z| = \frac{1}{2}$ under the transformation

$$w = \frac{z-2}{z-1}$$

~~Exercise 2D Page 48 Q's 5, 7, 8, 9, 12~~ Then mention standard tx's on pg 55

~~Q4 together then 6, 10, 18~~

~~then whatever time allows!~~

Ex 3H.

~~Eg21~~

Eg18

$$w = \frac{z-2}{z-1}$$

$$w(z-1) = z-2$$

$$wz - w = z - 2$$

$$wz - z = w - 2$$

$$z = \frac{w-2}{w-1}$$

$$|z| = \left|\frac{w-2}{w-1}\right|$$

$$|z| = \frac{|w-2|}{|w-1|}$$

when $|z|=1$ $\qquad 1 = \frac{|w-2|}{|w-1|}$

$|w-1| = |w-2|$ $\qquad$ perpendicular bisector of the points $(1,0)$ and $(2,0)$

ie a circle centre $(0,0)$ radius 1 in z plane becomes a straight line $\cancel{y} = \frac{3}{2}$ in w plane.

when $|z| = \frac{1}{2}$ $\qquad \frac{1}{2} = \frac{|w-2|}{|w-1|}$

$$|w-1| = 2|w-2|$$

let $w = u+iv$

$$|(u-1)+iv| = 2|(u-2)+iv|$$

$$\sqrt{(u-1)^2+v^2} = 2\sqrt{(u-2)^2+v^2}$$

$$(u-1)^2+v^2 = 4((u-2)^2+v^2)$$

$$u^2-2u+1+v^2 = 4u^2-16u+16+4v^2$$

$$3u^2-14u+3v^2 = -15$$

$\div 3$

$$u^2 - \frac{14u}{3} + v^2 = -5$$

$$\left(u-\frac{7}{3}\right)^2 + v^2 = -5 + \frac{49}{9}$$

$$\left(u-\frac{7}{3}\right)^2 + v^2 = \frac{4}{9}$$

$\left(u-\frac{7}{3}\right)^2 + v^2 = \left(\frac{2}{3}\right)^2$ $\qquad$ ie circle centre $\left(\frac{7}{3},0\right)$ radius $\frac{2}{3}$

ie a circle centre $(0,0)$ radius $\frac{1}{2}$ in z plane becomes a circle centre $\left(\frac{7}{3},0\right)$ radius $\frac{2}{3}$ in w plane.

Ex3H

① $w = z + 4 + 3i$

$z = w - (+4+3i)$

$|z| = |w - (+4+3i)|$

(a) $|z| = 1$

$\therefore \quad |w - (+4+3i)| = 1$ circle centre $(+4, +3)$ radius 1

(b) $\arg(z) = \frac{\pi}{2}$

$\arg\left(w - (4+3i)\right) = \frac{\pi}{2}$ half line from $(4,3)$ angle $\frac{\pi}{2}$

(c) lies on line $y = x$

loci of $|z-1| = |z+i|$

$|w - 4 - 3i - 1| = |w - 4 - 3i - i|$

$|w - (5+3i)| = |w - (4+4i)|$ perpendicular bisector between $(5,3)$ & $(4,4)$

①(c) alt: $z = w - 4 - 3i$

$x + iy = u + iv - 4 - 3i$

$x + iy = (u-4) + i(v-3)$

$x = u - 4$
$y = v - 3$

but $y = x$ $\therefore$ $v - 3 = u - 4$
$v = u - 1$

② translation $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ $w = z - 2 + 3i$

enlargement s.f. 4 $w = 4(z - 2 + 3i)$

$w = 4z - 8 + 12i$

③ $|z| = 2$

if $w = 3z + 2 - 5i$

$3z = w - 2 + 5i$

$z = \frac{w - 2 + 5i}{3}$

$|z| = \frac{|w - 2 + 5i|}{3}$

$2 = \frac{|w - 2 + 5i|}{3}$

$|w - (2 - 5i)| = 6$ circle centre $(2, -5)$ radius 6

④

$$w = 2z - 5 + 3i$$

$$2z = w + 5 - 3i$$

$$z = \frac{w}{2} + \frac{5}{2} - \frac{3i}{2}$$

$$z - \frac{5}{2} = \frac{w}{2} - \frac{3i}{2}$$

$$z - 2 - \frac{1}{2} = \frac{w}{2} - \frac{3i}{2}$$

$$z - 2 = \frac{w + 1 - 3i}{2}$$

$$|z - 2| = \frac{|w + 1 - 3i|}{2}$$

Now $|z-2| = 4$

$$4 = \frac{|w+1-3i|}{2}$$

$|w - (-1+3i)| = 8$ circle centre $(-1,3)$ radius 8

⑤

$$w = z - 1 + 2i$$

(a) $z - 1 = w - 2i$

$$|z-1| = |w - 2i|$$

Now $|z-1| = 3$

$|w - 2i| = 3$ circle centre $(0,2)$ radius 3

(b) $z = w + 1 - 2i$

Now $\arg(z - 1 + i) = \frac{\pi}{4}$

$$\arg(w + 1 - 2i - 1 + i) = \frac{\pi}{4}$$

$\arg(w - i) = \frac{\pi}{4}$ half line from $(0,1)$ angle $\frac{\pi}{4}$

5 (c) $z = w + 1 - 2i$

$x + iy = u + iv + 1 - 2i$

$x + iy = (u+1) + i(v-2)$

$x = u + 1$

$y = v - 2$

but $y = 2x$

so $v - 2 = 2(u+1)$

$v = 2u + 4$

6 (a) $w = \frac{1}{z}$

$z = \frac{1}{w}$

$|z| = \frac{1}{|w|}$

but $|z| = 2$

$|w| = \frac{1}{2}$ circle centre $(0,0)$ radius $\frac{1}{2}$

(b) $\arg z = \frac{\pi}{4}$

$\arg\left(\frac{1}{w}\right) = \frac{\pi}{4}$

$\arg 1 - \arg w = \frac{\pi}{4}$

$\arg w = \arg 1 - \frac{\pi}{4}$

$\arg w = -\frac{\pi}{4}$ half line centre $(0,0)$ angle $-\frac{\pi}{4}$

(c) $x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$

$x + iy = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$

(6)(c) cont'd $\quad x = \frac{u}{u^2+v^2} \quad y = -\frac{v}{u^2+v^2}$

but $y = 2x+1$

$$-\frac{v}{u^2+v^2} = 2\left(\frac{u}{u^2+v^2}\right) + 1$$

$$-v = 2u + u^2+v^2$$

$$u^2+2u+v^2+v = 0$$

$$(u+1)^2 + \left(v+\frac{1}{2}\right)^2 = 1+\frac{1}{4}$$

$$(u+1)^2 + \left(v+\frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$$

Circle centre $\left(-1, -\frac{1}{2}\right)$ radius $\frac{\sqrt{5}}{2}$ ← typo in ans back

(7) $\quad w = z^2$

(a) $|z| = 3$

$w = z^2$

$|w| = |z||z|$

$|w| = 9 \quad$ circle centre $(0,0)$ radius 9.

If $z = r(\cos\theta + i\sin\theta)$

then $w = (r(\cos\theta + i\sin\theta))^2$

$= r^2(\cos\theta + i\sin\theta)^2$

By DMTh $\quad w = r^2(\cos 2\theta + i\sin 2\theta)$

So as z moves through θ, w moves through 2θ.

(b) $z = x+0i$

$u+iv = (x+0i)^2$

$u+iv = x^2$

$v=0 \quad w = u \quad$ i.e. w lies on the +ve real axis $u \geq 0$.

(c) $z = 0+iy$

$u+iv = (0+iy)^2$

$u+iv = -y^2$

$v=0 \quad y^2 = -u$

$w = -u \quad$ i.e. w lies on $-$ve real axis $u \leq 0$ ← dodgy back of book

(8) (a) $|z+2i|<2$ ~~circle~~ inside circle centre $(0,-2)$ radius 2

(b) $w = z-2+5i$

$w = z-2+3i+2i$

$z+2i = w+2-3i$

$|z+2i| = |w+2-3i|$

$2 = |w+2-3i|$ circle centre $(-2,3)$ radius 2 interior.

(c) ~~$4z+4i = w-2$~~

$4z = w-2-4i$

$z = \dfrac{w-2-4i}{4}$

$\left|\dfrac{w-2-4i}{4}+2i\right|<2$

$\left|\dfrac{w-2-4i+8i}{4}\right|<2$

$|w-2+4i|<8$ circle centre $(2,-4)$ radius 8 interior

(d) $|zw+2iw| = 1$

$|w(z+2i)| = 1$

$|w||z+2i| = 1$

$|z+2i| = \dfrac{1}{|w|}$

if $|z+2i|<2$

then $\dfrac{1}{|w|}<2$

$\dfrac{1}{2}<|w|$

$|w|>\dfrac{1}{2}$ circle centre $(0,0)$ radius $\dfrac{1}{2}$ outside.

(9) $w = \dfrac{1}{2-z}$

$w(2-z) = 1$

$2w - wz = 1$

$2w - 1 = wz$

$z = \dfrac{2w-1}{w}$

$|z| = \dfrac{|2w-1|}{|w|}$

Now $|z| = 2$

$2|w| = |2w-1|$

$2|w| = |2(w-\tfrac{1}{2})|$

$2|w| = |2||w-\tfrac{1}{2}|$

$|w| = |w-\tfrac{1}{2}|$ perpendicular bisector between $(0,0)$ & $(\tfrac{1}{2},0)$

ie $u = \dfrac{1}{4}$

(10). $w = \dfrac{16}{z}$

(a) $z = \dfrac{16}{w}$

when $|z-4| = 4$

$\left|\dfrac{16}{w} - 4\right| = 4$

$\left|\dfrac{16-4w}{w}\right| = 4$

$\cancel{|4|(4-w)|} = 4|w|$

$\cancel{|4||4-w|}$

$|-4||w-4| = 4|w|$

$4|w-4| = 4|w|$

$|w-4| = |w|$ perp bisector between $(0,0)$ & $(4,0)$

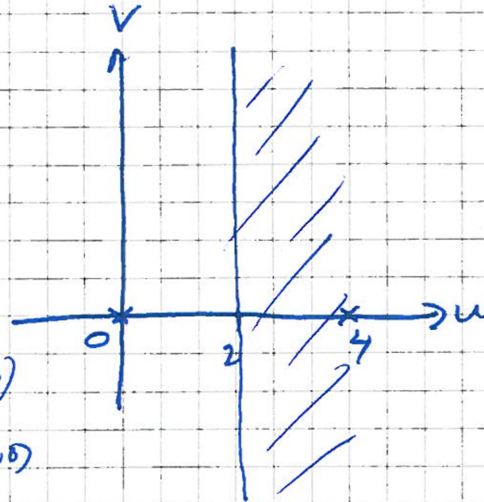
$u = 2$

(b)

$|z-4| < 4$

$|w-4| < |w|$

Dist of P from $(4,0)$ is less than dist from $(0,0)$



11 $w = \frac{3}{2-z}$

$w(2-z) = 3$

$2w - wz = 3$

$wz = 2w - 3$

$z = \frac{2w-3}{w}$

$x+iy = \frac{2(u+iv)-3}{u+iv}$

$x+iy = \frac{(2u-3)+i2v}{u+iv} \times \frac{u-iv}{u-iv}$

$x+iy = \frac{(2u-3)u + i2uv - iv(2u-3) + 2v^2}{u^2+v^2}$

$x+iy = \frac{2u^2-3u+2v^2}{u^2+v^2} + i\left[\frac{3v}{u^2+v^2}\right]$

$x = \frac{2u^2-3u+2v^2}{u^2+v^2} \qquad y = \left[\frac{3v}{u^2+v^2}\right]$

Now $2y = x$

$2\left[\frac{3v}{u^2+v^2}\right] = \frac{2u^2-3u+2v^2}{u^2+v^2}$

$6v = 2u^2 - 3u + 2v^2$

$2u^2 + 2v^2 - 3u - 6v = 0$

$\div 2 \quad u^2 - \frac{3}{2}u + v^2 - 3 = 0$

$\left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{9}{16} + \frac{9}{4}$

$\left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \left(\frac{\sqrt{45}}{4}\right)^2 = \left(\frac{3\sqrt{5}}{4}\right)^2$

$\therefore$ circle centre $\left(\frac{3}{4}, \frac{3}{2}\right)$ radius $\frac{3\sqrt{5}}{4}$ As required

(12) (a) $w = -\frac{iz+i}{z+1}$

$w(z+1) = -iz+i$

$wz+w = -iz+i$

$\cancel{wz+iz = i}$

$wz+iz = i-w$

$z(w+i) = i-w$

$z = \frac{i-w}{i+w}$

~~using~~ $x^2+y^2=1 \Rightarrow |z|=1$

$|z| = \left|\frac{i-w}{i+w}\right|$

$1 = \frac{|i-w|}{|i+w|}$

$|w+i| = |-1(w-i)|$

$|w+i| = |-1||w-i|$

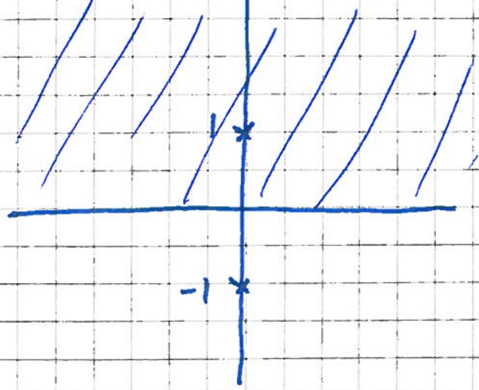
$|w+i| = |w-i|$ perp bisector between $(0,1)$ and $(0,-1)$

eqⁿ of l $v=0$

(b) if $|z| \le 1$ then $\frac{|i-w|}{|i+w|} \le 1$

$|w-i| \le |w+i|$

dist of P from ~~(1,0)~~ $(0,1)$ ~~<~~ $\le$ dist of P from $(0,-1)$



solution is opposite side, but I'm not convinced.

12 c) $x^2+y^2=4 \Rightarrow |z|=2$

$$2=\left|\frac{i-w}{i+w}\right|$$

$$2|w+i|=|w-i|$$

$$2|u+i(v+1)|=|u+i(v-1)|$$

$$4\left(u^2+(v+1)^2\right)=u^2+(v-1)^2$$

$$4u^2+4v^2+8v+4-u^2-v^2+2v-1=0$$

$$3u^2+3v^2+10v+3=0$$

$$u^2+v^2+\frac{10}{3}v=-1$$

$$u^2+\left(v+\frac{5}{3}\right)^2=-1+\frac{25}{9}=\frac{16}{9}$$

$$u^2+\left(v+\frac{5}{3}\right)^2=\left(\frac{4}{3}\right)^2 \quad \therefore$$

13. $w=\dfrac{4z-3i}{z-1}$

$$w(z-1)=4z-3i$$

$$wz-w=4z-3i$$

$$wz-4z=w-3i$$

$$z(w-4)=w-3i$$

$$|z|=\frac{|w-3i|}{|w-4|}$$

$$|z|=3$$

$$3|w-4|=|w-3i|$$

$$3|(u-4)+iv|=|u+i(v-3)|$$

$$9\left((u-4)^2+v^2\right)=u^2+(v-3)^2$$

$$9u^2-72u+144+9v^2-u^2-v^2+6v-9=0$$

$$\cancel{8u^2-72u+8v^2=-135}$$

$$\div 8 \quad \cancel{u^2-9u+v^2=\frac{-135}{8}}$$

$$\cancel{\left(u-\frac{9}{2}\right)^2+v^2=\frac{-135}{8}+\frac{81}{4}=\frac{27}{8}}$$

$$8u^2-72u+8v^2+6v=-135$$

$$\div 8 \quad u^2-9u+v^2+\frac{3}{4}v=\frac{-135}{8}$$

$$\left(u-\frac{9}{2}\right)^2+\left(v+\frac{3}{8}\right)^2=\frac{-135}{8}+\frac{81}{4}+\frac{9}{64}$$

13 contd $\left(u-\frac{9}{2}\right)^2+\left(v+\frac{3}{8}\right)^2=\frac{225}{64}=\left(\frac{15}{8}\right)^2$

$\therefore$ circle centre $\left(\frac{9}{2},-\frac{3}{8}\right)$ radius $\frac{15}{8}$

14 (a) $w=\frac{1}{z+i}$

$w(z+i)=1$

$wz+wi=1$

$wz=1-wi$

$z=\frac{1-iw}{w}$

when $z=x+0i$ $\quad w=u+iv$

$iy+x=\frac{1-i(u+iv)}{u+iv}$

$iy+x=\frac{1\cancel{*}-iu+v}{u+iv}$

$iy+x=\frac{(1+v)-iu}{u+iv}\times\frac{u-iv}{u-iv}$

$iy+x=\frac{u(1+v)-iu^2-iv(1+v)-uv}{u^2+v^2}$

$iy+x=\frac{u+\cancel{uv}-i(u^2+v^2+v)-\cancel{uv}}{u^2+v^2}$

$x=\frac{u}{u^2+v^2}\qquad y=\frac{u^2+v^2+v}{u^2+v^2}$

Now $y=0\qquad u^2+v^2+v=0$

$u^2+\left(v+\frac{1}{2}\right)^2=\frac{1}{4}=\left(\frac{1}{2}\right)^2$ Circle centre $\left(0,-\frac{1}{2}\right)$ radius $\frac{1}{2}$

(b) $x=4\qquad \frac{u}{u^2+v^2}=4$ * Sol^n suggests $\frac{u}{u^2+v^2}=2$?

$u=4u^2+4v^2$

$u^2-\frac{1}{4}u+v^2=0$

$\left(u-\frac{1}{8}\right)^2+v^2=\frac{1}{64}=\left(\frac{1}{8}\right)^2$ Circle centre $\left(\frac{1}{8},0\right)$ radius $\frac{1}{8}$ *

(15). $w = z + \frac{4}{z}$

$$w = \frac{z^2+4}{z}$$

$$w = \frac{(x+iy)^2+4}{(x+iy)}$$

$$w = \frac{x^2-y^2+2xyi+4}{(x+iy)} \times \frac{x-iy}{x-iy}$$

$$w = \frac{x^3-xy^2+2x^2yi+4x-iyx^2+iy^3+2xy^2-4yi}{x^2+y^2}$$

$$w = \frac{x^3-xy^2+4x+2xy^2}{x^2+y^2} + i\left[\frac{2x^2y-x^2y+y^3-4y}{x^2+y^2}\right]$$

$$w = \frac{x(x^2+y^2+4)}{x^2+y^2} + iy\left[\frac{x^2+y^2-4}{x^2+y^2}\right]$$

Now $|z|=2 \Leftrightarrow x^2+y^2=4$

$$w = x\frac{(4+4)}{4} + iy\left[\frac{4-4}{4}\right]$$

$$w = 2x + 0i$$

$$u+iv = 2x+0i$$

$$u=2x, \quad v=0$$

Now $|z|=2$ circle centre 0,0 radius 2, so $-2 \leq x \leq 2$

$\therefore$ $2(-2) \leq u \leq 2(2)$

$$-4 \leq u \leq 4$$

$\therefore$ $|z|=2$ mapped onto real axis in w-plane with interval $[-4,4]$

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$$w = \frac{1}{z+3}$$

$$w(z+3) = 1$$

$$wz + 3w = 1$$

$$wz = 1 - 3w$$

$$z = \frac{1-3w}{w}$$

$$z = \frac{1-3(u+iv)}{u+iv}$$

$$z = \frac{1-3u-3vi}{u+iv} \times \frac{u-iv}{u-iv}$$

$$z = \frac{u - 3u^2 - 3\cancel{uvi} - iv + 3\cancel{uvi} - 3v^2}{u^2+v^2}$$

$$\underset{x+iy}{z} = \frac{u-3u^2-3v^2}{u^2+v^2} + i\left[\frac{\cancel{3uv} - v}{u^2+v^2}\right]$$

$$x = \frac{u-3u^2-3v^2}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2}$$

$$2x - 2y + 7 = 0$$

$$2\left[\frac{u-3u^2-3v^2}{u^2+v^2}\right] - 2\left[\frac{-v}{u^2+v^2}\right] + 7 = 0$$

$$2u - 6u^2 - 6v^2 + 2v + 7u^2 + 7v^2 = 0$$

$$u^2 + v^2 + 2v + 2u = 0$$

$$(u+1)^2 + (v+1)^2 = 2$$

circle centre $(-1,-1)$ radius $\sqrt{2}$