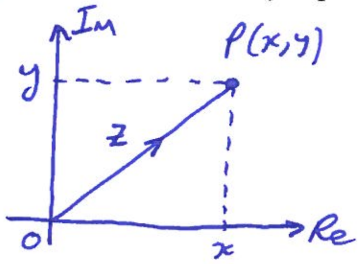
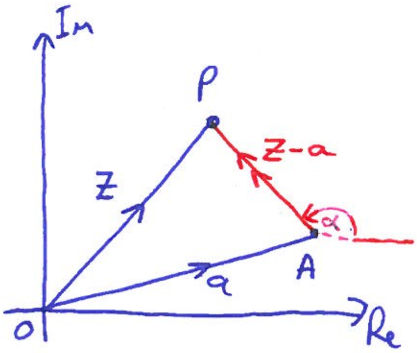
Loci in the Argand diagram

Consider the complex number $z = x + iy$ represented by the vector $\overrightarrow{OP}$, where P is the point (x, y).



If z varies subject to some given condition, then the corresponding set of points in the Argand diagram is called the locus of P. Since many loci can be defined using distances or angles, equations of loci in the Argand diagram often involve moduli or arguments of complex variables.

Consider now the introduction of a constant complex number, a, represented by the vector $\overrightarrow{OA}$:



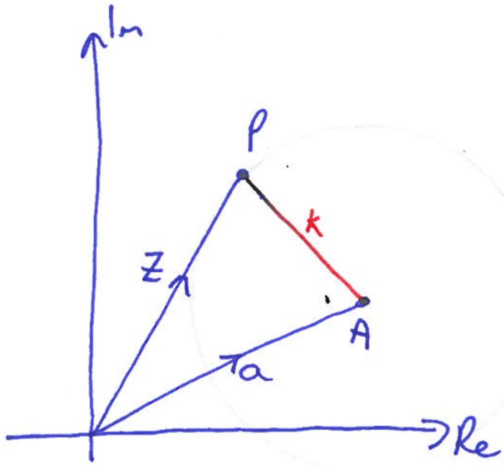
$$\overrightarrow{AP} = -\overrightarrow{OA} + \overrightarrow{OP}$$
$$= -a + z$$
$$= z - a$$

$|z-a|$ = distance AP

$\arg(z-a) = \alpha$ = angle $\overrightarrow{AP}$ makes with +ve Real axis.

Loci of the form $|z - a| = k$

It follows from the above Argand diagram that if $|z - a| = k$, where k is a positive constant, then the distance from P to A is constant. **Thus the locus so described will be a circle with centre at the point A and radius k.**



This can be shown algebraically:

$\cancel{|x+iy-a|=k}$ if $a = x_0 + iy_0$

$\cancel{|(x-a)+iy|=k}$ Then...

$\cancel{(x-a)^2+y^2=k^2}$

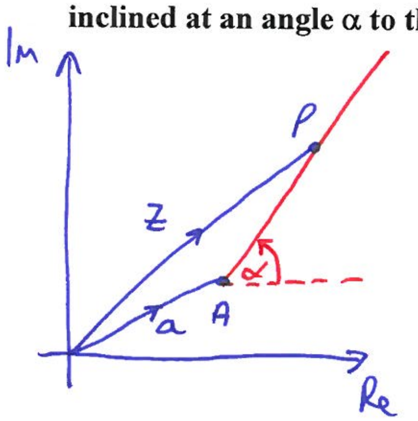
$$|x+iy-(x_0+iy_0)| = k$$
$$|(x-x_0)+i(y-y_0)| = k$$
$$(x-x_0)^2+(y-y_0)^2 = k^2$$

which is equation of circle, centre (x_0, y_0) and radius k.

Loci of the form $\arg(z-a)=\alpha$

If $\arg(z-a)=\alpha$, where α is constant and $-\pi<\alpha\le\pi$, then the direction of $\overrightarrow{AP}$ is constant. **Thus the locus so described will be a *half-line* with end-point A, inclined at an angle α to the real axis.**

a half-line, or ray, ~~...~~ has one end, but stretches off to infinity in the other direction. Like a ray of light which starts in a definite place but never ends.



Algebraically

$\arg((x-x_0)+i(y-y_0))=\alpha$

$\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right)=\alpha$

$\frac{y-y_0}{x-x_0}=\tan\alpha$

$y-y_0=\tan\alpha(x-x_0)$

Represents eq^n of half line starting (x_0,y_0) and gradient $\tan\alpha$.

Eg10 ~~Eg12~~ Sketch the loci defined by the equations (a) $|z|=2$, (b) $|z-3+2i|=5$

Eg11 ~~Eg13~~ Sketch the loci defined by the equations (a) $\arg(z-1-2i)=\frac{\pi}{4}$,

(b) $\arg(z+2)=-\frac{2}{3}\pi$

Exercise ~~2C~~ 3F Page ~~41~~ 49 Q's 1 & 2 ~~left columns and 1(h)~~

Other Complex Loci

When dealing with the remaining standard complex loci, it is likely that an algebraic approach will be necessary along with an understanding of the geometry.

Eg12 ~~Eg14~~ Sketch the locus of the point P(x, y) representing the complex number z = x + iy, given that $|z-1|=|z+i|$. Determine the Cartesian equation of the locus.

Ex3F Q3,4,5

Eg13 ~~Eg15~~ . Find the Cartesian equation of the locus of the point P(x, y) representing the complex number z, given that $2|z-3i|=|z|$. Show that the locus is a circle, giving its centre and radius.

In general, if a point P moves so that the ratio of its distances from two fixed points A and B is constant, then the locus of P is a circle. This locus is referred to Apollonius' circle and is represented on the Argand diagram by an equation of the form

$|z-a|=k|z-b|$ or $\left|\frac{z-a}{z-b}\right|=k$ where $k\neq1$

As seen in Eg14, when $k=1$, the equation represents the perpendicular bisector of the line joining A to B.

~~Exercise 2C Page 42 Q's 3 & 4 left columns~~

~~Then 6, 7, 8, 10, 11, 13~~

Ex3F Q6, 8→16

Equations involving $\arg\left(\dfrac{z-a}{z-b}\right)$ are more difficult to interpret.

Eg14 ~~Eg16~~ Sketch the locus of z where $\arg\left(\dfrac{z-3}{z+1}\right)=\dfrac{\pi}{4}$

Eg15 ~~Eg17~~ Sketch the locus of z where $\arg\left(\dfrac{z+2}{z-i}\right)=\dfrac{\pi}{3}$

These loci can also be determined algebraically, but can become quite involved.

Eg16 ~~Eg18~~ Do Eg~~16~~ 14 again, but using algebra to determine the Cartesian equation of the loci.

Usually the geometric approach is enough unless the equation is specifically requested.

Ex3F Q7

~~Exercise 2C Page 42 Q5 all, then Q's 9 & 14~~

Inequalities involving the modulus or argument of a complex variable can be used to represent regions of an Argand diagram.

Eg17 ~~Eg19~~ Shade in separate Argand diagrams the regions in which

(a) $|z-1-i|\le 3$

(b) $\dfrac{\pi}{4}\le \arg(z-2)\le\dfrac{\pi}{3}$

Eg18 ~~Eg20~~ If z_1 and z_2 are complex numbers, show geometrically that

(a) $|z_1-z_2|\ge|z_2|-|z_1|$

(b) $|z_1+z_2|\ge\big||z_1|-|z_2|\big|$

~~Exercise 2D Page 48 Q's 1, 2, 3~~

Ex3G.

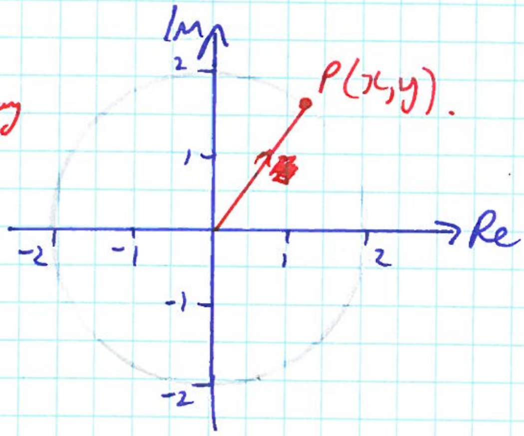
~~Eg 12~~ Eg 16 (a) $|z| = 2$

Algebra not necessary

$|x+iy| = 2$

$x^2+y^2=4$

circle centre $(0,0)$, centre 2



(b) $|z-3+2i| = 5$

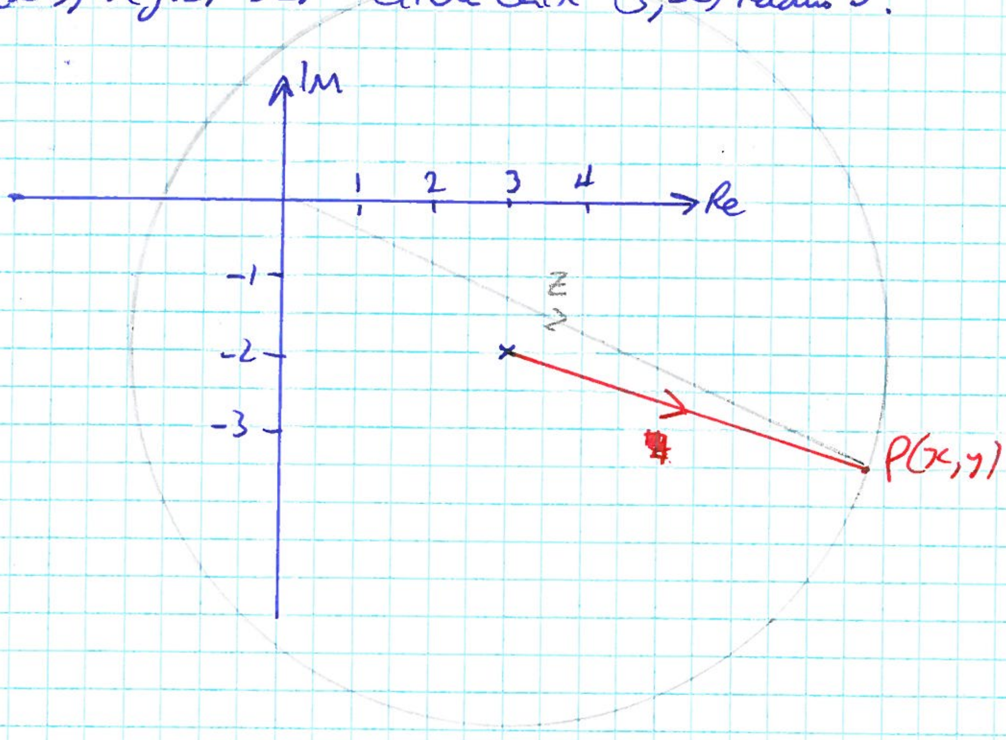
$|z-(3-2i)| = 5$ distance of P from the point representing $3-2i$ is 5 units

ie circle centre $(3,-2)$ ~~centre~~ radius 5

or algebraically

$|(x-3)+i(y+2)| = 5$

$(x-3)^2+(y+2)^2 = 25$ circle centre $(3,-2)$ Radius 5.



~~Eg 13~~ (a) $\arg(z-1-2i)=\frac{\pi}{4}$

Eg 11

So $z-1-2i = z-(1+2i)$

Now if A is the point representing $(1+2i)$, $\arg(z-1-2i)$ is the angle AP makes with the real axis.

Hence $\arg(z-1-2i)=\frac{\pi}{4}$ represents the half line with endpoint $(1,2)$, inclined at an angle $\frac{\pi}{4}$ (45°) to real axis.

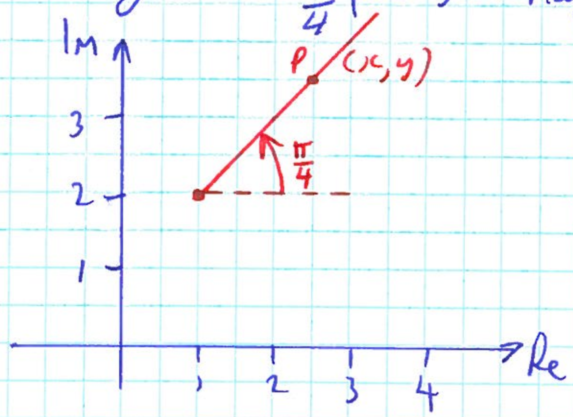
or algebraically

$$\arg\left((x-1)+i(y-2)\right)=\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{y-2}{x-1}\right)=\frac{\pi}{4}$$

$$\frac{y-2}{x-1}=\tan\frac{\pi}{4}$$

$y-2=\tan\frac{\pi}{4}(x-1)$ half-line from $(1,2)$ gradient 1.



~~Eg 13 (b)~~ Eg 11 $\arg(z+2) = -\frac{2}{3}\pi$

$\arg(z-(-2+0i)) = -\frac{2}{3}\pi$ A point $(-2,0)$, inclined at $-\frac{2}{3}\pi$ to real axis.

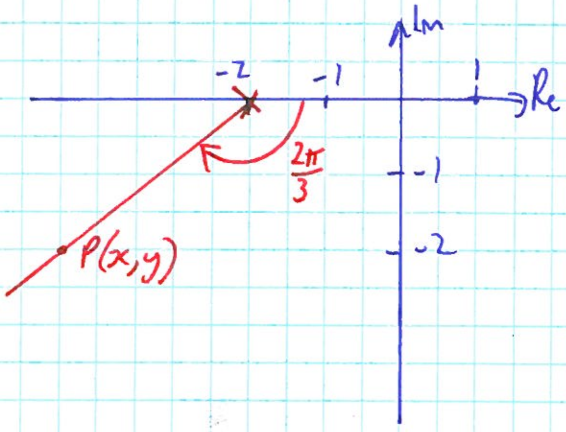
Algebraically $\arg\left((x+2)+iy\right) = -\frac{2}{3}\pi$

$\tan^{-1}\left(\frac{y}{x+2}\right) = -\frac{2}{3}\pi$

$\frac{y}{x+2} = \tan\left(-\frac{2}{3}\pi\right)$

$y = \tan\left(-\frac{2}{3}\pi\right)(x+2)$ line starting @ $(-2,0)$ gradient $\sqrt{3}$

$y = \sqrt{3}(x+2)$



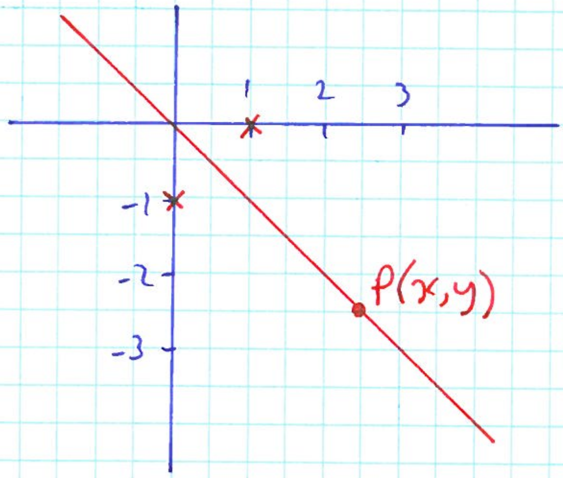
~~Eg 4~~ Eg 2 $|z-1| = |z+i|$

$|z-(1+0i)| = |z-(0-i)|$

distance between $P(x,y)$ and $(1,0)$ = distance between $P(x,y)$ and $(0,-1)$

ie P is equidistant from $(1,0)$ and $(-1,0)$

perpendicular bisector.



algebraically $|(x-1)+iy| = |x+i(y+1)|$

$$(x-1)^2+y^2 = x^2+(y+1)^2$$

$$x^2-2x+1+y^2 = x^2+y^2+2y+1$$

$$-2x = 2y$$

$$y = -x$$ ⇐ cartesian eq^n of locus P.

~~Eg 5~~ Eg 3 $2|z-3i| = |z|$ ⟸ The distance of P from $(0,0)$ is twice the distance of P from $(0,3)$

$$2|x+i(y-3)| = |x+iy|$$

$$4[x^2+(y-3)^2] = x^2+y^2$$

$$4x^2+4y^2-24y+36 = x^2+y^2$$

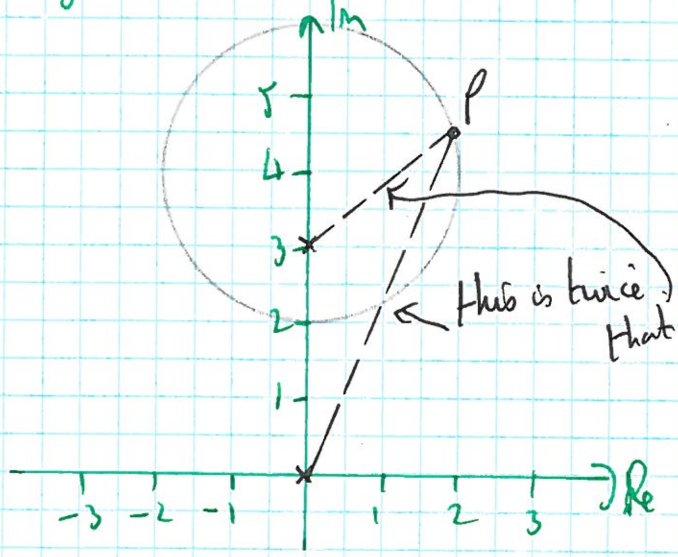
$$3x^2+3y^2-24y+36=0$$

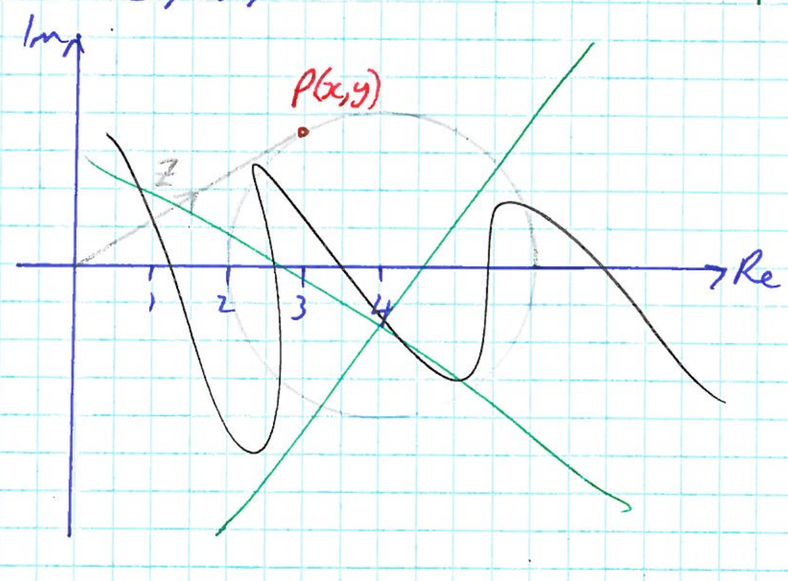
$\div 3$ $$x^2+y^2-8y = \cancel{36}-12$$

$$x^2+(y-4)^2 = \cancel{36}-12+16$$

$$x^2+(y-4)^2 = 4$$

circle centre $(0,4)$, radius 2.





~~Eg16~~ Eg14 $\quad \arg\left(\frac{z-3}{z+1}\right) = \frac{\pi}{4}$

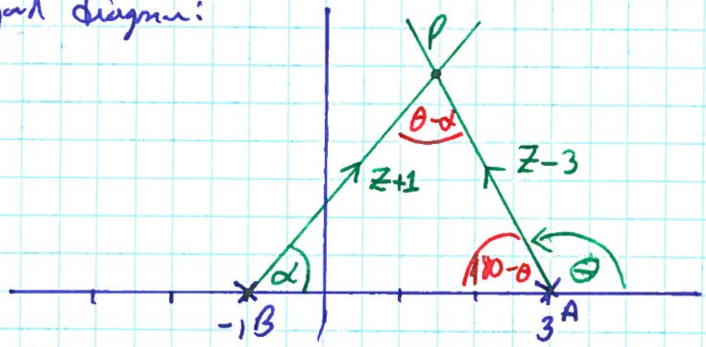
From FP1: $\arg\left(\frac{z-3}{z+1}\right) = \arg(z-3) - \arg(z+1)$

$\therefore \arg(z-3) - \arg(z+1) = \frac{\pi}{4}$ —— (1)

From Eg 13, $\arg(z-3)$ represents the half line starting at $\overset{A}{(3,0)}$ inclined at an angle, say θ to the ~~hori~~ positive real axis

Similarly $\arg(z+1)$ represents the half line starting at $\overset{B}{(-1,0)}$ inclined at an angle say α to the positive real axis.

Consider the argand diagram:



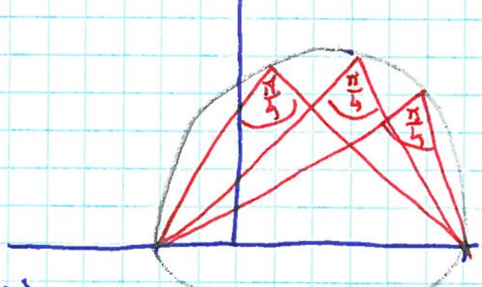
By geometry $\hat{APB} = 180 - \alpha - (180 - \theta) = \theta - \alpha$

But from (1) $\quad \underset{\Downarrow}{\arg(z-3)} - \underset{\Downarrow}{\arg(z+1)} = \frac{\pi}{4}$

$\theta - \alpha = \frac{\pi}{4}$

$\therefore \hat{APB} = \text{constant} = \frac{\pi}{4}$

From GCSE circle theorems, angles in the same segment of a circle are equal. This implies that P lies on the arc of a circle which passes through A and B.



You cannot always tell from this however where the centre of the circle lies or whether the arc lies above, or below the real axis.

Eg16 contd. ~~However in this example the centre can be determined using symmetry and more circle theorem stuff...~~

~~$\frac{\pi}{2} = 90°$, so angle at circumference is a right angle which mean AB is diameter and $|AB| = 4$ ∴ centre is at $(1,0)$. ∴ cartesian eqⁿ $(x-1)^2+y^2=4$~~

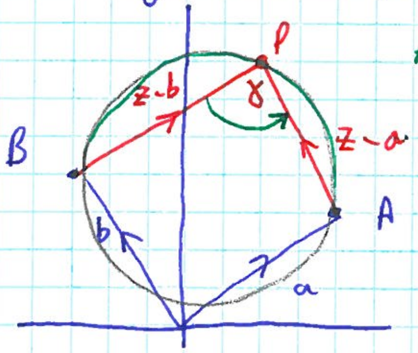
To determine which arc of the circle the locus is defined for you must consider in which direction the angle turns to go from $\overrightarrow{BP}$ to $\overrightarrow{AP}$.

If the given angle is +ve, this turn needs to be anti-clockwise, a negative angle would require a clockwise turn.

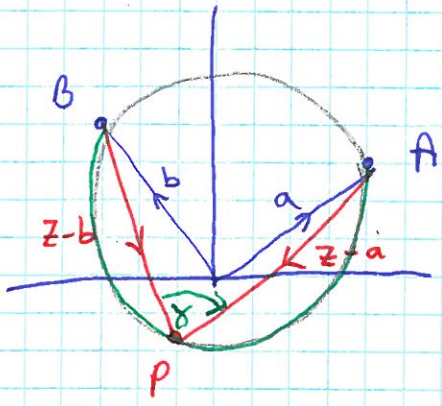
Consider the following:

If $\quad \arg\left(\frac{z-a}{z-b}\right) = \gamma$

$\arg(z-a) - \arg(z-b) = \gamma$

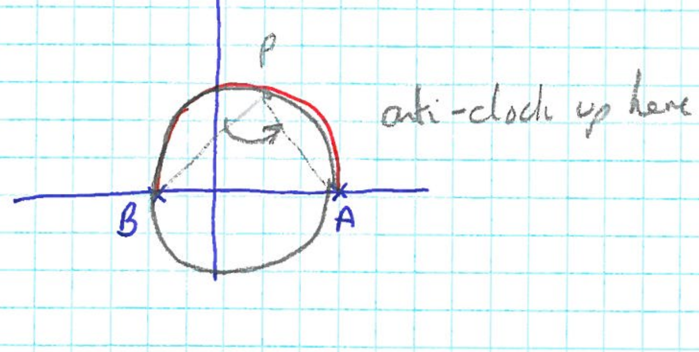


turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ is anticlockwise so $\gamma > 0$ and required arc is top of circle.

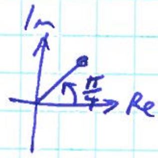


turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ is clockwise so $\gamma < 0$ and required arc is bottom of circle

In this particular example $\gamma = \frac{\pi}{4}$ = anti-clockwise turn



Gy16 (algebraic) cont'd



Now to determine which arc....

For $\frac{z-3}{z+1}$ to have and argument $\frac{\pi}{4}$, real + imaginary parts must both be +ve

ie $\frac{x^2-2x-3+y^2}{(x+1)^2+y^2} > 0$ and $\frac{4y}{(x+1)^2+y^2} > 0$

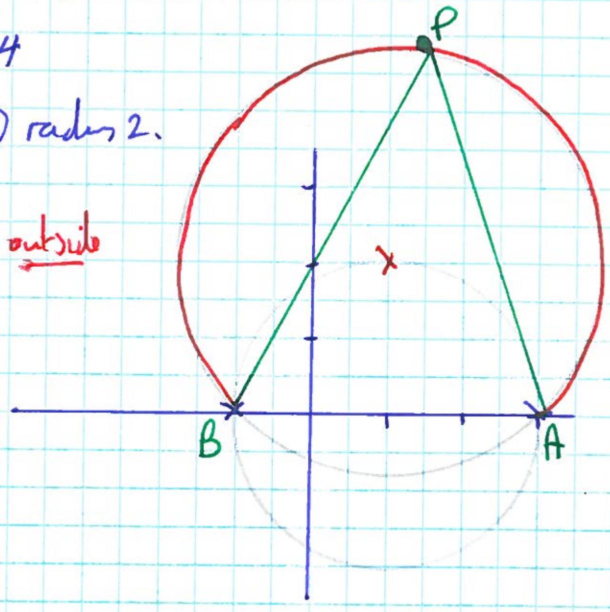
~~so~~ $x^2-2x-3+y^2 > 0$ $\qquad y > 0$

$(x-1)^2+y^2 > 3+1$

$(x-1)^2+y^2 > 4$

circle centre (1,0) radius 2.

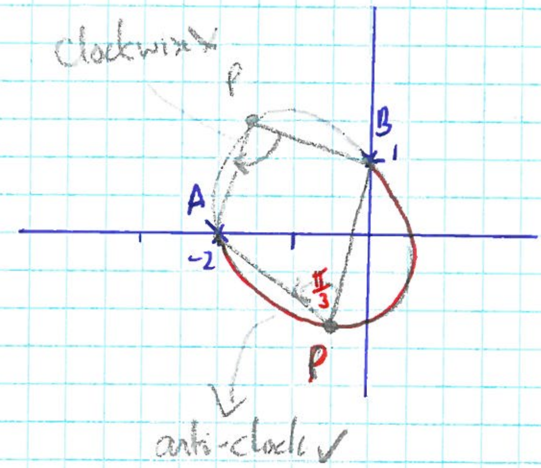
arc which falls outside this circle,



~~Eg 14~~ Eg 15 $\arg\left(\frac{z+2}{z-i}\right) = \frac{\pi}{3}$

arc of circle passing through A $(-2,0)$ and B$(0,1)$

Turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ must be anti-clockwise



arc identified in red.

These examples can both be considered algebraically...

~~Eg 16~~ Eg 16 $\arg\left(\frac{z-3}{z+1}\right) = \frac{\pi}{4}$

$$\arg\left(\frac{(x-3)+iy}{(x+1)+iy}\right) = \frac{\pi}{4}$$

$$\arg\left[\frac{(x-3)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}\right] = \frac{\pi}{4}$$

$$\arg\left[\frac{(x-3)(x+1) - iy(x-3) + iy(x+1) + y^2}{(x+1)^2+y^2}\right] = \frac{\pi}{4}$$

$$\arg\left[\frac{x^2-2x-3+y^2}{(x+1)^2+y^2} + \frac{i4y}{(x+1)^2+y^2}\right] = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{4y}{x^2-2x-3+y^2}\right) = \frac{\pi}{4}$$

$$\frac{4y}{x^2-2x-3+y^2} = 1$$

$$4y = x^2-2x-3+y^2$$

$$x^2-2x+y^2-4y = 3$$

$$(x-1)^2+(y-2)^2 = 3+1+4$$

$$(x-1)^2+(y-2)^2 = 8$$

$\therefore$ locus is arc of circle centre $(1,2)$ radius $2\sqrt{2}$

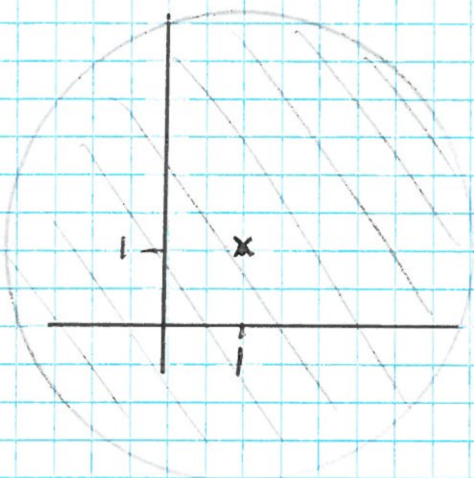
~~Eg 19~~ (a) $|z - 1 - i| < 3$

Eg 17

$|z - (1+i)| < 3$

So $|z-(1+i)| = 3$ represents circle centre $(1,1)$ radius 3.

$\therefore$ $|z-(1+i)| < 3$ represents the interior and circumference of this circle

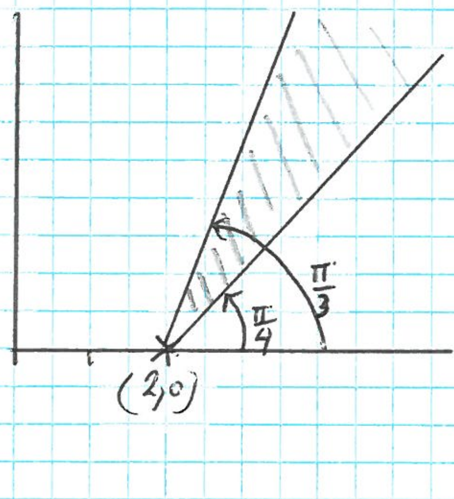


(b) $\frac{\pi}{4} \leq \arg(z-2) \leq \frac{\pi}{3}$

Now $\arg(z-2) = \frac{\pi}{3}$ represents half line from $(2,0)$, angle $\frac{\pi}{3}$ to x-axis

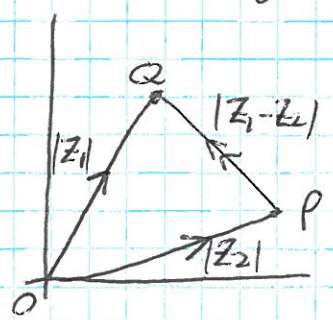
and $\arg(z-2) = \frac{\pi}{4}$ " " " " " " $\frac{\pi}{4}$ " " "

hence $\frac{\pi}{4} \leq \arg(z-2) \leq \frac{\pi}{3}$ represents these two lines and the angle between them



~~Eg 10~~ Eg18 (a) $|z_1 - z_2| \geq |z_2| - |z_1|$

to get this, get both numbers from origin



For ΔOPQ $\quad OP \leq OQ + QP$

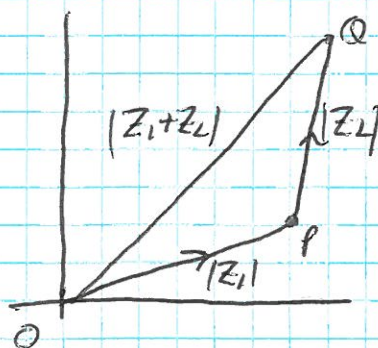
$$|z_2| \leq |z_1| + |z_1 - z_2|$$

$$|z_2| - |z_1| \leq |z_1 - z_2|$$

$$\therefore |z_1 - z_2| \geq |z_2| - |z_1|$$

(b) $|z_1 + z_2| \geq \big|\,|z_1| - |z_2|\,\big|$

to get this have z_2 carry off z_1



For ΔOPQ: $\quad OP \leq OQ + PQ$

$$|z_1| \leq |z_1 + z_2| + |z_2|$$

$$|z_1| - |z_2| \leq |z_1 + z_2|$$

$$|z_1 + z_2| \geq |z_1| - |z_2| \quad \text{— (1)}$$

Similarly: $\quad PQ \leq OQ + OP$

$$|z_2| \leq |z_1 + z_2| + |z_1|$$

$$|z_1 + z_2| \geq |z_2| - |z_1| \quad \text{— (2)}$$

Now $\big|\,|z_1| - |z_2|\,\big| = |z_2| - |z_1|$ and $|z_1| - |z_2|$

So $|z_1 + z_2| \geq \big|\,|z_1| - |z_2|\,\big|$

Ex3F

① (a) $|z| = 6$

$|x + iy| = 6$

$x^2 + y^2 = 6^2$ circle centre $(0,0)$ radius 6

(b) $|z| = 10$

$|x + iy| = 10$

$x^2 + y^2 = 100$ circle centre $(0,0)$ radius 10

(c) $|z - 3| = 2$

$|x + iy - 3| = 2$

$(x-3)^2 + y^2 = 4$ circle centre $(3,0)$ radius 2

(d) $|z + 3i| = 3$

$|x + (y+3)i| = 3$

$x^2 + (y+3)^2 = 9$ circle centre $(0,-3)$ radius 3

(e) $|z - 4i| = 5$

$|x + (y-4)i| = 5$

$x^2 + (y-4)^2 = 25$ circle centre $(0,4)$ rad 5

(f) $|z + 1| = 1$

$|(x+1) + iy| = 1$

$(x+1)^2 + y^2 = 1$ circle centre $(-1,0)$ rad 1

(g) $|z - 1 - i| = 5$

$|(x-1) + i(y-1)| = 5$

$(x-1)^2 + (y-1)^2 = 25$ circle centre $(1,1)$ rad 5

1 (h) $|z+3+4i|=4$

$|(x+3)+i(y+4)|=4$

$(x+3)^2+(y+4)^2=16$ circle centre $(-3,-4)$ rad 4

(i) $|z-5+6i|=5$

$|(x-5)+i(y+6)|=5$

$(x-5)^2+(y+6)^2=25$ circle centre $(5,-6)$ rad 5

(j) $|2z+6-4i|=6$

$|2x+2yi+6-4i|=6$

$|2(x+3)+2i(y-2)|=6$

$(2(x+3))^2+(2(y-2))^2=36$

$4\left[(x+3)^2+(y-2)^2\right]=36$

$(x+3)^2+(y-2)^2=9$ circle centre $(-3,2)$ rad 3

(k) $|3z-9-6i|=12$

$|3x+3yi-9-6i|=12$

$|3(x-3)+3i(y-2)|=12$

$(3(x-3))^2+(3(y-2))^2=144$

$9\left[(x-3)^2+(y-2)^2\right]=144$

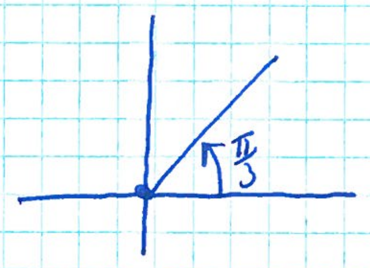
$(x-3)^2+(y-2)^2=\frac{144}{9}=16$ circle centre $(3,2)$ rad 4

① (l) $|3z-9-6i| = 9$

Same LHS as (k)

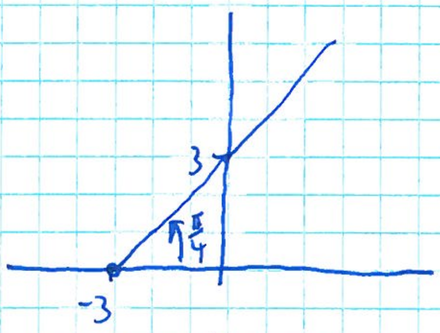
$(x-3)^2 + (y-2)^2 = \frac{81}{9} = 9$ circle centre $(3,2)$ rad 3

② (a) $\arg z = \frac{\pi}{3}$ half line from $(0,0)$ angle $\frac{\pi}{3}$



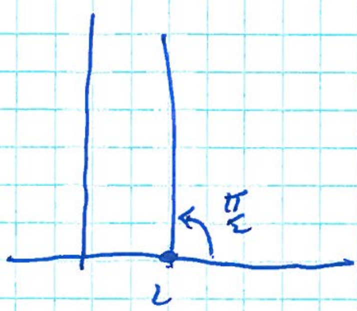
(b) $\arg(z+3) = \frac{\pi}{4}$

$\arg(z-(-3+0i)) = \frac{\pi}{4}$ half line from $(-3,0)$ angle $\frac{\pi}{4}$



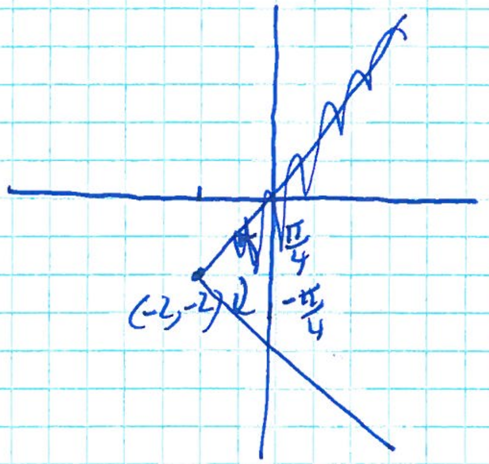
(c) $\arg(z-2) = \frac{\pi}{2}$

$\arg(z-(2+0i)) = \frac{\pi}{2}$ half line from $(2,0)$ angle $\frac{\pi}{2}$



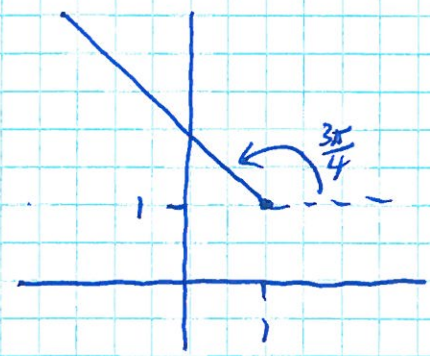
② (d) $\arg(z+2+2i) = -\frac{\pi}{4}$

$\arg(z-(-2-2i)) = -\frac{\pi}{4}$ half line from $(-2,-2)$ angle $-\frac{\pi}{4}$



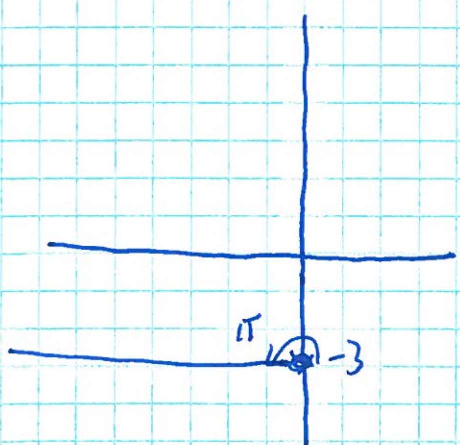
(e) $\arg(z-1-i) = \frac{3\pi}{4}$

$\arg(z-(1+i)) = \frac{3\pi}{4}$ half line from $(1,1)$ angle $\frac{3\pi}{4}$



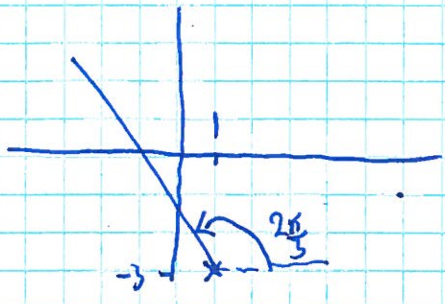
(f) $\arg(z+3i) = \pi$

$\arg(z-(0-3i)) = \pi$ half line from $(0,-3)$ angle π



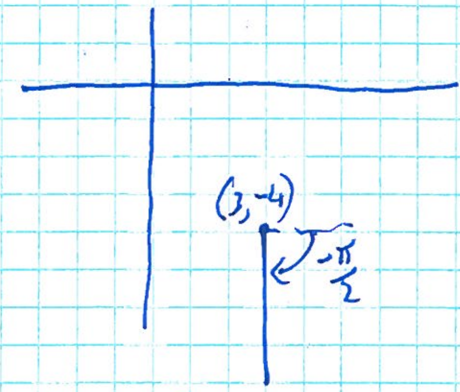
② (g) $\arg(z-1+3i) = \frac{2\pi}{3}$

$\arg(z-(1-3i)) = \frac{2\pi}{3}$ half line from $(1,-3)$ angle $\frac{2\pi}{3}$



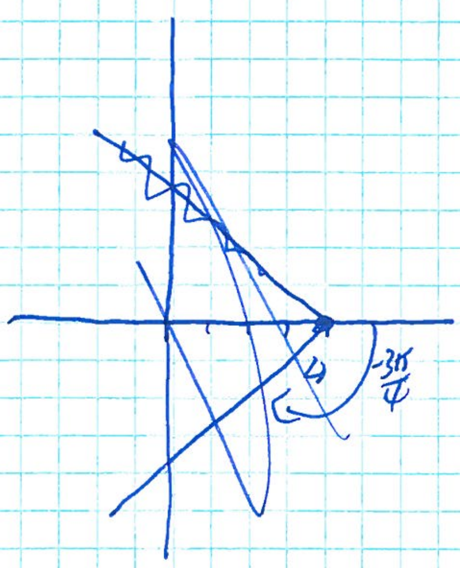
(h) $\arg(z-3+4i) = \cancel{3}\ -\frac{\pi}{2}$

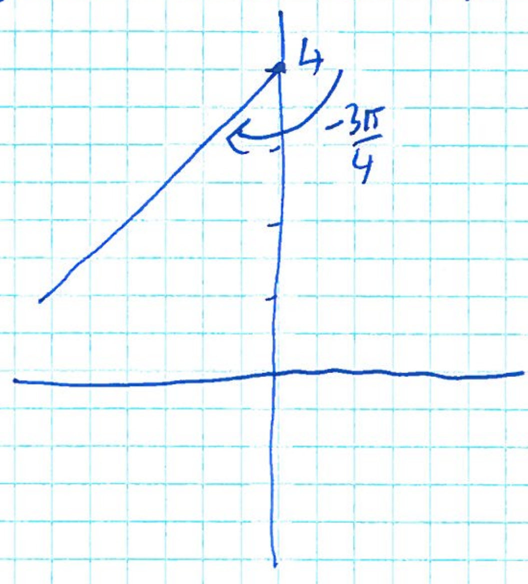
$\arg(z-(3-4i)) = \cancel{4}\ -\frac{\pi}{2}$ half line from $(3,-4)$ angle $-\frac{\pi}{2}$



(i) $\arg(z-4i) = -\frac{3\pi}{4}$

$\arg(z-(0+4i)) = -\frac{3\pi}{4}$ half line from $(0,4)$ angle $-\frac{3\pi}{4}$





③ (a) $|z-6| = |z-2|$

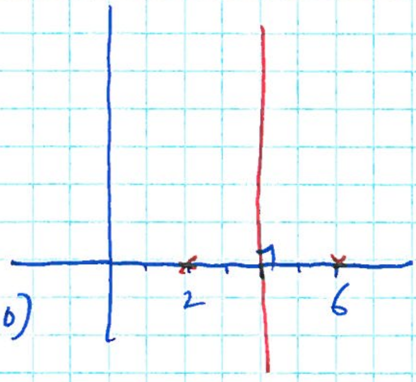
$|(x-6)+iy| = |(x-2)+iy|$

$(x-6)^2 + \cancel{y^2} = (x-2)^2 + \cancel{y^2}$

$\cancel{x^2} - 12x + 36 = \cancel{x^2} - 4x + 4$

$8x = 32$

$x = 4$ perp bisect bet $(2,0)$ & $(6,0)$



(b) $|z+8| = |z-4|$

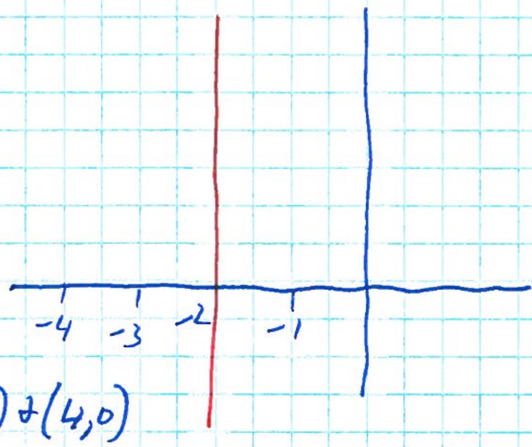
$|(x+8)+iy| = |(x-4)+iy|$

$(x+8)^2 + \cancel{y^2} = (x-4)^2 + \cancel{y^2}$

$\cancel{x^2} + 16x + 64 = \cancel{x^2} - 8x + 16$

$24x = -48$

$x = -2$ perp bis bet $(-8,0)$ & $(4,0)$



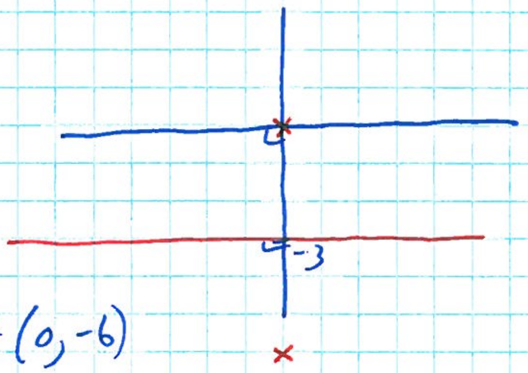
(c) $|z| = |z+6i|$

$|x+iy| = |x+i(y+6)|$

$\cancel{x^2} + y^2 = \cancel{x^2} + (y+6)^2$

$\cancel{y^2} = \cancel{y^2} + 12y + 36$

$y = -3$ perp bis bet $(0,0)$ & $(0,-6)$



(d) $|z+3i| = |z-8i|$

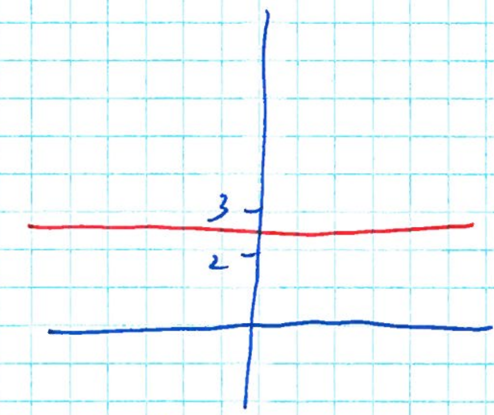
$|x+i(y+3)| = |x+i(y-8)|$

$\cancel{x^2} + (y+3)^2 = \cancel{x^2} + (y-8)^2$

$\cancel{y^2} + 6y + 9 = \cancel{y^2} - 16y + 64$

$22y = 55$

$y = 2.5$ perp bis bet $(0,-3)$ & $(0,8)$



③ (e) $|z-2-2i| = |z+2+2i|$

$$|(x-2)+i(y-2)| = |(x+2)+i(y+2)|$$

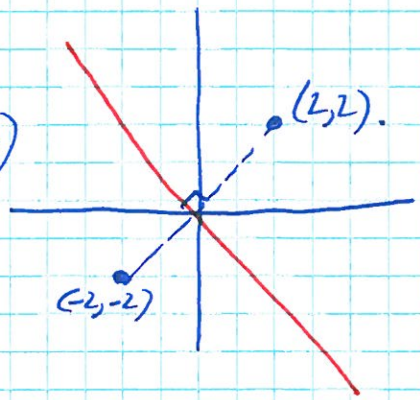
$$(x-2)^2+(y-2)^2 = (x+2)^2+(y+2)^2$$

$$\cancel{x^2}-4x+\cancel{4}+\cancel{y^2}-4y+\cancel{4} = \cancel{x^2}+4x+\cancel{4}+\cancel{y^2}+4y+\cancel{4}$$

$$-8x = 8y$$

$$y = -x$$

perp bis between $(2,2)$ and $(-2,-2)$



(f) $|z+4+i| = |z+4+6i|$

$|z-(-4-i)| = |z-(-4-6i)|$ perp bis between $(-4,-1)$ and $(-4,-6)$

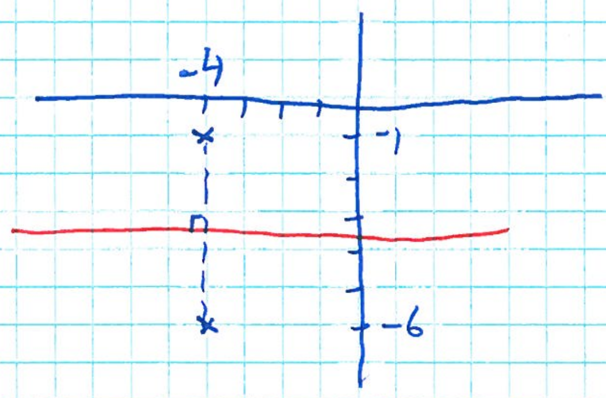
$$|(x+4)+i(y+1)| = |(x+4)+i(y+6)|$$

$$(\cancel{x+4})^2+(y+1)^2 = (\cancel{x+4})^2+(y+6)^2$$

$$\cancel{y^2}+2y+1 = \cancel{y^2}+12y+36$$

$$10y = -35$$

$$y = -3.5$$



③(g) $|z+3-5i| = |z-7-5i|$

$|z-(-3+5i)| = |z-(7+5i)|$ so perp bisect bet $(-3,5)$ & $(7,5)$

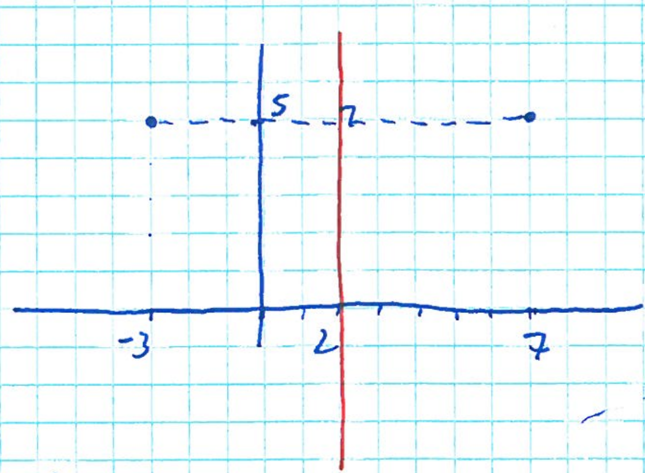
$|(x+3)+i(y-5)| = |(x-7)+i(y-5)|$

$(x+3)^2+(y-5)^2 = (x-7)^2+(y-5)^2$

$\cancel{x^2}+6x+9+\cancel{y^2}-\cancel{10y}+\cancel{25} = \cancel{x^2}-14x+49+\cancel{y^2}-\cancel{10y}+\cancel{25}$

$20x = 40$

$x = 2$



(h) $|z+4-2i| = |z-8+2i|$

$|z-(-4+2i)| = |z-(8-2i)|$ perp bis bet $(-4,2)$ and $(8,-2)$

$|(x+4)+i(y-2)| = |(x-8)+i(y+2)|$

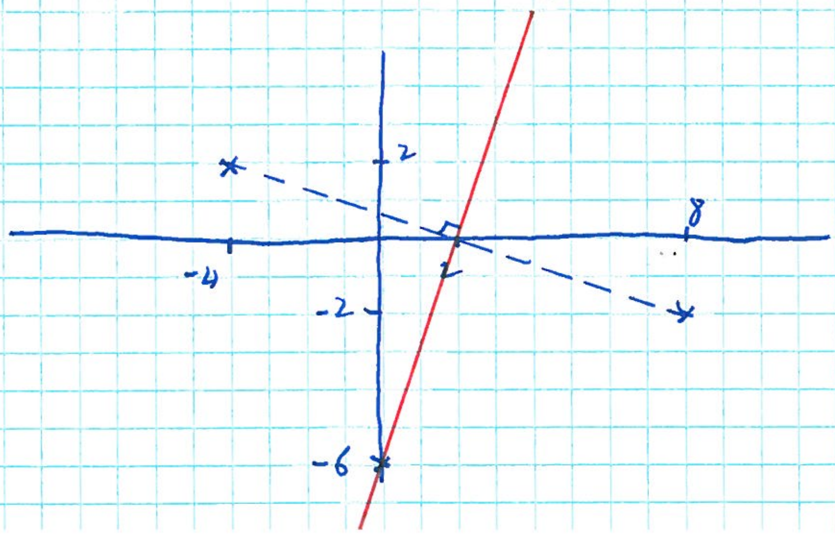
$(x+4)^2+(y-2)^2 = (x-8)^2+(y+2)^2$

$(x+4)^2-(x-8)^2 = (y+2)^2-(y-2)^2$

$x^2+8x+16-(x^2-16x+64) = y^2+4y+4-(y^2-4y+4)$

$24x-48 = 8y$

$y = 3x-6$



③ (i) $\dfrac{|z+3i|}{|z-6i|} = 1$ * typo in book solution are for $\dfrac{|z+3|}{|z-6i|} = 1$.

$|z+3i| = |z-6i|$

$|z-(0-3i)| = |z-(0+6i)|$ perp bis bet $(0,-3)$ & $(0,6)$

$|x+i(y+3)| = |x+i(y-6)|$

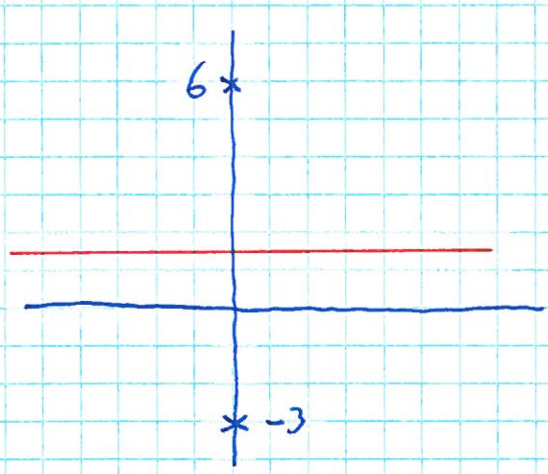
$x^2+(y+3)^2 = x^2+(y-6)^2$

$(y+3)^2-(y-6)^2 = 0$

$y^2+6y+9-(y^2-12y+36) = 0$

$18y - 27 = 0$

$y = +\dfrac{27}{18} = +1.5$



(j) $\dfrac{|z+6-i|}{|z-10-5i|} = 1$

$|z-(-6+i)| = |z-(+10+5i)|$ perp bis bet $(-6,1)$ & $(10,5)$

$|(x+6)+i(y-1)| = |(x-10)+i(y-5)|$

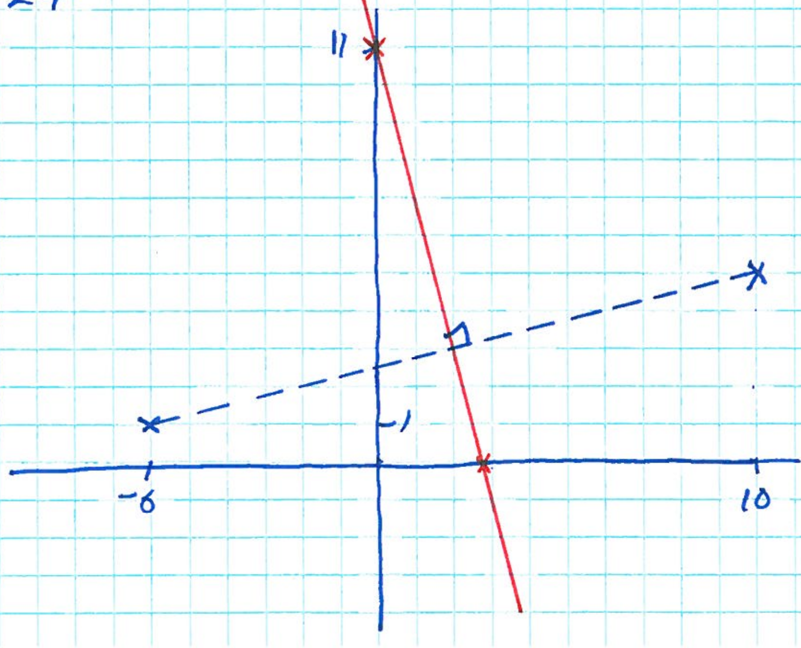
$(x+6)^2-(x-10)^2 = (y-5)^2-(y-1)^2$

$x^2+12x+36-(x^2-20x+100) = y^2-10y+25-(y^2-2y+1)$

$32x-64 = -8y+24$

$8y = -32x+88$

$y = -4x+11$



③ (k) $|z+7+2i| = |z-4-3i|$

$|z-(-7-2i)| = |z-(4+2i)|$ perp bis bet $(-7,-2)$ & $(4,2)$

$|(x+7)+i(y+2)| = |(x-4)+i(y-3)|$

$(x+7)^2-(x-4)^2 = (y-3)^2-(y+2)^2$

$x^2+14x+49-(x^2-8x+16) = y^2-6y+9-(y^2+4y+4)$

$22x+33 = -10y+5$

$10y = -22x-28$

$y = -2.2x-2.8$

$|z-(-1+6i)| = |-1(z-2-3i)|$

(l) $\cancel{|z+1|}$ $|z+1-6i| = |2+3i-z|$ $|z-(-1+6i)| = |-1||z-(2+3i)|$

$\uparrow 1$

$|(x+1)^2+i(y-6)| = |(2-x)^2+i(3-y)|$ perp bis bet $(-1,6)$ & $(2,3)$.

$(x+1)^2-(2-x)^2 = (3-y)^2-(y-6)^2$

$x^2+2x+1-(4-4x+x^2) = 9-6y+y^2-(y^2-12y+36)$

$6x-3 = 6y-27$

$6y = 6x+24$

$y = x+4$

4 (a) $z - z^* = 0$

$x + iy - (x - iy) = 0$

$2yi = 0$

$y = 0.$

(b) $z + z^* = 0$

$x + iy + (x - iy) = 0$

$2x = 0$

$x = 0$

5 (a) $|2 - z| = 3$

$|-1(z-2)| = 3$

$|-1||z-2| = 3 \quad |-1| = 1$

$|z - (2+0i)| = 3$ circle centre $(2,0)$ radius 3.

$(x-2)^2 + y^2 = 9$

(b) $|5i - z| = 4$

$|-1(z - 5i)| = 4$

$|-1|\,|z - (0+5i)| = 4$ circle centre $(0,5)$ radius 4

$x^2 + (y-5)^2 = 16$

(c) $|3 - 2i - z| = 3$

$|-1(z - 3 + 2i)| = 3$

$|-1||z - (3-2i)| = 3$ circle centre $(3,-2)$ radius 3

$(x-3)^2 + (y+2)^2 = 9$

⑥ (a) $|z+3| = 3|z-5|$

$$|(x+3)+iy| = 3|(x-5)+iy|$$

$$(x+3)^2+y^2 = 9[(x-5)^2+y^2]$$

$$x^2+6x+9+y^2 = 9[x^2-10x+25+y^2]$$

$$8x^2-96x+8y^2+216=0$$

$\div 8$

$$x^2-12x+y^2+27=0$$

$$(x-6)^2+y^2+27-36=0$$

$(x-6)^2+y^2=9$ $\therefore$ circle centre $(6,0)$ radius 3.

(b) $|z-3| = 4|z+1|$

$$|(x-3)+iy| = 4|(x+1)+iy|$$

$$(x-3)^2+y^2 = 16[(x+1)^2+y^2]$$

$$x^2-6x+9+y^2 = 16[x^2+2x+1+y^2]$$

$$15x^2+38x+7+15y^2=0$$

$\div 15$ $x^2+\frac{38}{15}x+y^2+\frac{7}{15}=0$

$$\left(x+\frac{19}{15}\right)^2+y^2+\frac{7}{15}-\frac{19^2}{15^2}=0$$

$\left(x+\frac{19}{15}\right)^2+y^2=\frac{256}{225}$ $\therefore$ circle centre $\left(\frac{-19}{15},0\right)$ rad $\frac{16}{15}$.

(6) (c) $|z-i| = 2|z+i|$

$|x+i(y-1)| = 2|x+i(y+1)|$

$x^2+(y-1)^2 = 4[x^2+(y+1)^2]$

$x^2+y^2-2y+1 = 4[x^2+y^2+2y+1]$

$3x^2+3y^2+10y+3 = 0$

$x^2+y^2+\frac{10}{3}y+1 = 0$

$x^2+\left(y+\frac{5}{3}\right)^2+1-\frac{25}{9} = 0$

$x^2+\left(y+\frac{5}{3}\right)^2 = \frac{16}{9}$ circle centre $\left(0, -\frac{5}{3}\right)$ rad $\frac{4}{3}$

(d) $|z+2-7i| = 2|z-10+2i|$

$|(x+2)+i(y-7)| = 2|(x-10)+i(y+2)|$

$(x+2)^2+(y-7)^2 = 4[(x-10)^2+(y+2)^2]$

$x^2+4x+4+y^2-14y+49 = 4[x^2-20x+100+y^2+4y+4]$

$3x^2-84x+3y^2+30y+363 = 0$

$x^2-28x+y^2+10y+121 = 0$

$(x-14)^2+(y+5)^2+121-196-25 = 0$

$(x-14)^2+(y+5)^2 = 100$ circle centre $(14, -5)$ rad 10

6 (e) $|z+4-2i| = 2|z-2-5i|$

$|(x+4)+i(y-2)| = 2|(x-2)+i(y-5)|$

$(x+4)^2+(y-2)^2 = 4\left[(x-2)^2+(y-5)^2\right]$

$x^2+8x+16+y^2-4y+4 = 4\left[x^2-4x+4+y^2-10y+25\right]$

$3x^2-24x+3y^2-36y+96=0$

$x^2-8x+y^2-12y=-32$

$(x-4)^2+(y-6)^2 = -32+16+36$

$(x-4)^2+(y-6)^2 = 20$ circle centre $(4,6)$ rad $2\sqrt{5}$

(f) $|z| = 2|2-z|$

$|x+iy| = 2|(2-x)-iy|$

$x^2+y^2 = 4\left[(2-x)^2+y^2\right]$

$x^2+y^2 = 4\left[4-4x+x^2+y^2\right]$

$3x^2-16x+3y^2+16=0$

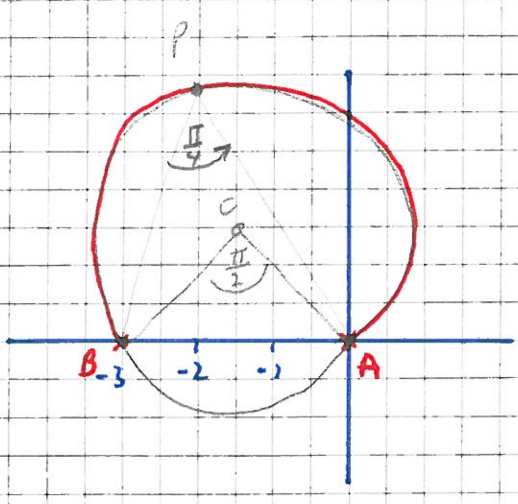
$x^2-\frac{16x}{3}+y^2 = -\frac{16}{3}$

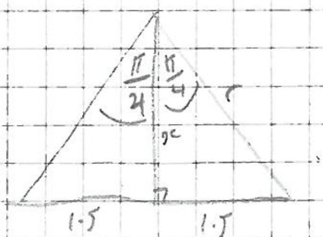
$\left(x-\frac{8}{3}\right)^2+y^2 = -\frac{16}{3}+\frac{64}{9}$

$\left(x-\frac{8}{3}\right)^2+y^2 = \frac{16}{9}$ circle centre $\left(\frac{8}{3},0\right)$ rad $\frac{4}{3}$

7 (a) $\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$ arc of circle passing through $A(0,0)$ and $B(-3,0)$

Turn from $\vec{BP}$ to $\vec{AP}$ must be anti-clockwise





$\frac{1.5}{x} = \tan\frac{\pi}{4}$

$x = \frac{1.5}{\tan\frac{\pi}{4}} = 1.5$

$\frac{1.5}{r} = \sin\frac{\pi}{4}$

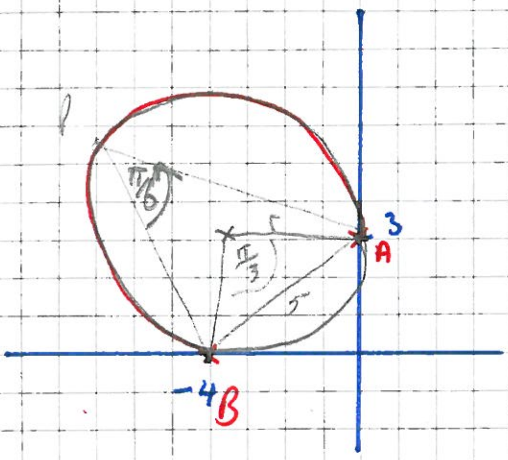
$r = \frac{1.5}{\sin\frac{\pi}{4}} = \frac{3}{2} \times \frac{\sqrt{2}}{1} = \frac{3\sqrt{2}}{2}$

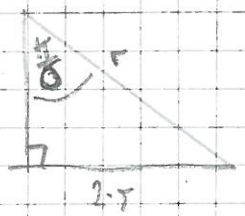
$\therefore$ circle centre $\left(-\frac{3}{2}, \frac{3}{2}\right)$ radius $\frac{3\sqrt{2}}{2}$

$\left(x+\frac{3}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{9}{2}$

(7)(b) $\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$ arc of circle passing through $A(0,3)$ & $B(-4,0)$

Turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ must be anti-clockwise



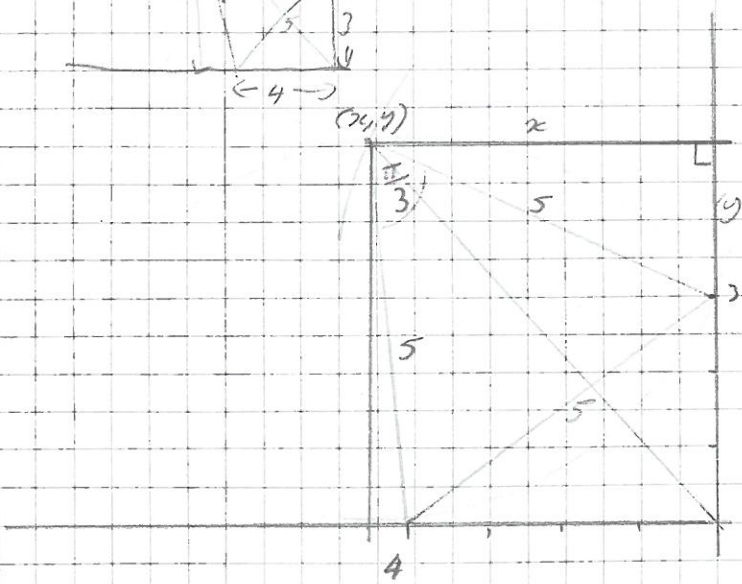


$\frac{2.5}{r} = \sin\frac{\pi}{6}$

~~$\frac{5}{2r} = \frac{\sqrt{3}}{2}$~~

~~$r = \frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$~~

$r = 5$



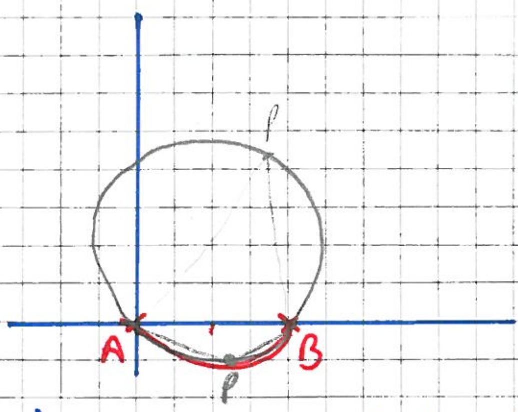
$x^2 + (y-3)^2 = 25$

$(x-4)^2 + y^2 = 25$

PAH!

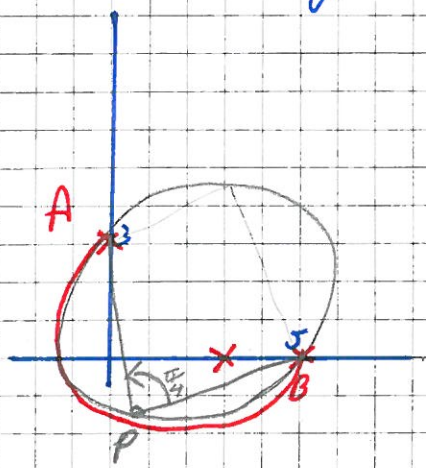
7 (c) $\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$ arc of circle passing thru' A(0,0) & B(2,0)

Turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ must be anti-clock



(d) $\arg\left(\frac{z-3i}{z-5}\right) = \frac{\pi}{4}$ arc thru' A(0,3) and B(5,0)

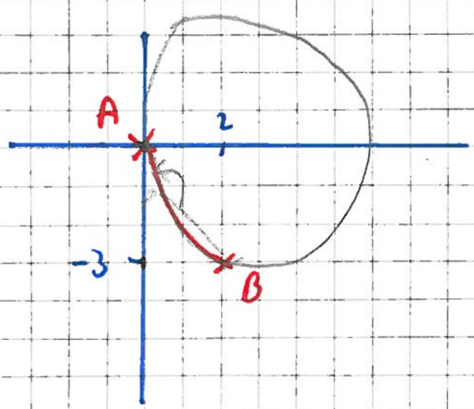
Turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ anti-clock



7 e) $\arg z - \arg(z-2+3i) = \frac{\pi}{3}$

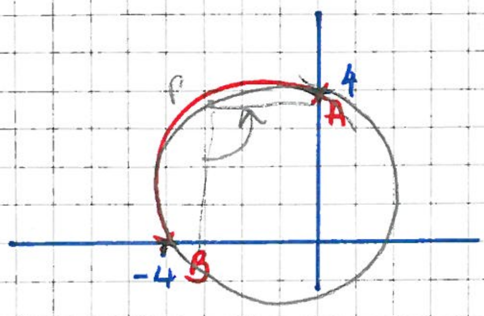
$\arg z - \arg(z-(2-3i)) = \frac{\pi}{3}$ Arc passing thru A(0,0) and B(2,−3)

turn from $\overrightarrow{BP}$ to $\overrightarrow{AP}$ must be anti-clock



⑦(f) $\arg\left(\frac{z-4i}{z+4}\right)=\frac{\pi}{2}$ Arc passing through $A(0,4)$ & $B(-4,0)$

$\overrightarrow{BP}$ to $\overrightarrow{AP}$ anti-clock



(8) $|z| = 5$ circle centre $(0,0)$ radius 5. $x^2 + y^2 = 25$ — (1)

$\arg(z+4) = \frac{\pi}{2}$

$\arg(z - (-4+0i)) = \frac{\pi}{2}$ half line from $(-4,0)$ angle $\frac{\pi}{2}$

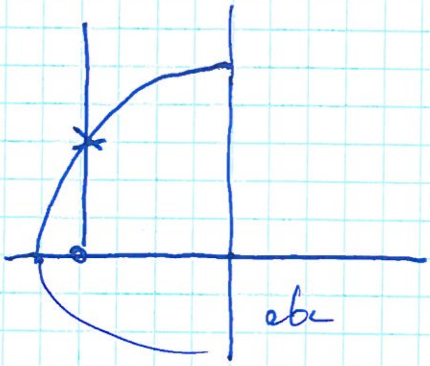
$\arg((x+4) + iy) = \frac{\pi}{2}$

$\tan^{-1}\left(\frac{y}{x+4}\right) = \frac{\pi}{2}$

$\frac{y}{x+4} = \infty$

$\therefore \quad x + 4 = 0$

$x = -4$



in (1) $(-4)^2 + y^2 = 25$

$y^2 = 9$

$y = \pm 3$

$y = +3$

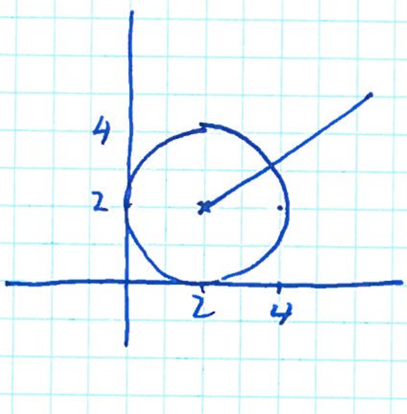
$\therefore \quad z = -4 + 3i$

(9)(a) $|z - 2 - 2i| = 2$

$|z - (2+2i)| = 2$ circle centre $(2,2)$ radius 2

$(x-2)^2 + (y-2)^2 = 4$ — (1)

(b) $\arg(z - (2+2i)) = \frac{\pi}{6}$ half line from $(2,2)$ angle $\frac{\pi}{6}$



$\arg((x-2) + i(y-2)) = \frac{\pi}{6}$

$\tan^{-1}\left(\frac{y-2}{x-2}\right) = \frac{\pi}{6}$

$\frac{y-2}{x-2} = \frac{1}{\sqrt{3}}$

$\sqrt{3}(y-2) = (x-2)$ — (2)

9 contd From 2 $(x-2)^2 = 3(y-2)^2$ — 3

in 1 $3(y-2)^2 + (y-2)^2 = 4$

$4(y-2)^2 = 4$

$(y-2)^2 = 1$

$y-2 = \pm 1$

$y = 2 \pm 1$

from diagram $y = 3$

in 3 $(x-2)^2 = 3(3-2)^2$

$(x-2)^2 = 3$

$x-2 = \pm\sqrt{3}$

$x = 2 \pm \sqrt{3}$

from diagram $x = 2+\sqrt{3}$

$\therefore\ Z = (2+\sqrt{3}) + 3i$

10 (a) $|Z-2i| = |Z-8i|$

$|Z-(0+2i)| = |Z-(0+8i)|$ so perp bisector between $(0,2)$ and $(0,8)$

eq^n $y = 5$ — 1

(b) $\arg(Z-2-i) = \frac{\pi}{4}$

$\arg(Z-(2+i)) = \frac{\pi}{4}$ half line from $(2,1)$ angle $\frac{\pi}{4}$

(c) $\arg((x-2)+i(y-1)) = \frac{\pi}{4}$

$\tan^{-1}\left(\frac{y-1}{x-2}\right) = \frac{\pi}{4}$

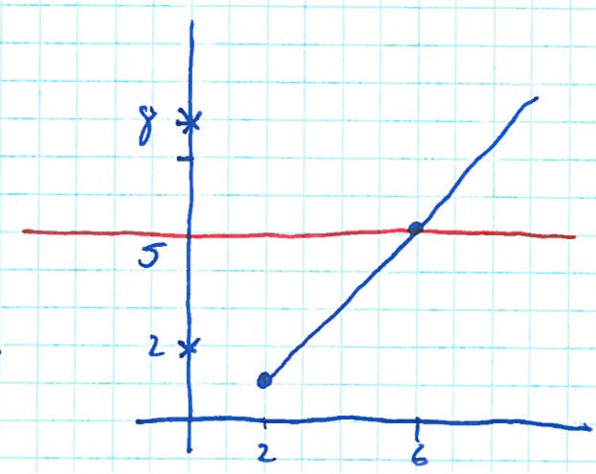
$y-1 = x-2$

$y = x-1$ — 2

equating 1 & 2 $x-1 = 5$

$x = 6$

$\therefore\ Z = 6+5i$



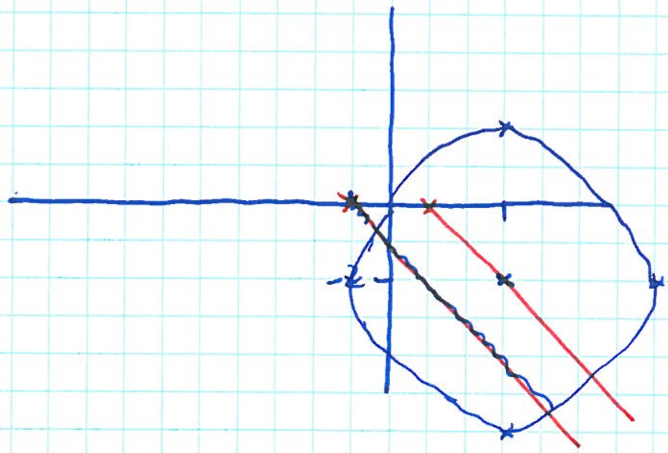
(11) (a) $|z-3+2i|=4$

$|z-(3-2i)|=4$ circle centre $(3,-2)$ radius 4

eq^n $(x-3)^2+(y+2)^2=16$ — (1)

(b) $\arg(z-1)=-\frac{\pi}{4}$ half line from $(+1,0)$ angle $-\frac{\pi}{4}$

$\arg((x-1)+iy)=-\frac{\pi}{4}$



$\tan^{-1}\left(\frac{y}{x-1}\right)=-\frac{\pi}{4}$

$\frac{y}{x-1}=-1$

$y=-x+1$ — (2)

in (1) $(x-3)^2+(-x+1+2)^2=16$

$x^2-6x+9+9-6x+x^2=16$

$2x^2-12x+2=0$

$x^2-6x+1=0$.

$(x-3)^2-8=0$

$(x-3)^2=8$

$x-3=\pm 2\sqrt{2}$

$x=3\pm 2\sqrt{2}$

from diagram $x=3+2\sqrt{2}$

in (2) $y=-(3+2\sqrt{2})+1=-2-2\sqrt{2}$

$\therefore$ $z=(3+2\sqrt{2})+(-2-2\sqrt{2})i$

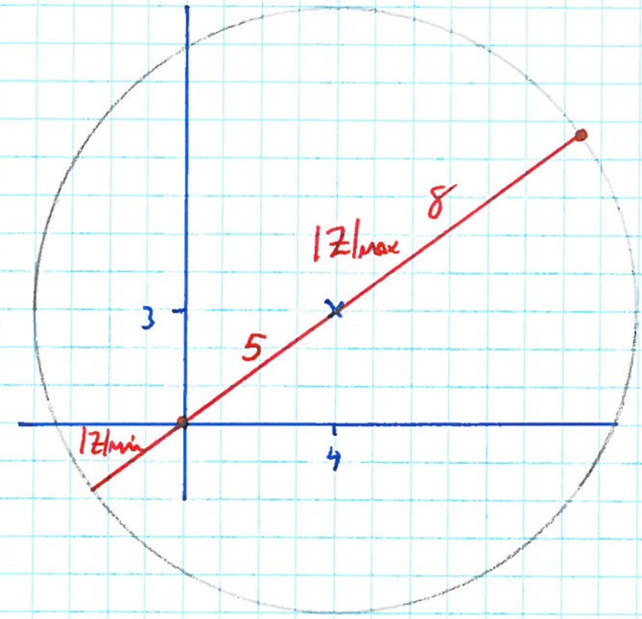
(12) (a) $|z-4-3i|=8$

$$|(x-4)+i(y-3)|=8$$

$$(x-4)^2+(y-3)^2=64$$

(b) circle centre $(4,3)$ radius 8.

(c)

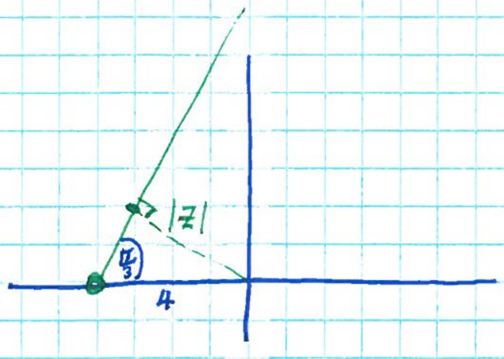


(d) $|z|_{max} = 5+8 = 13$

$|z|_{min} = 8-5 = 3$

(13) $\arg(z+4) = \frac{\pi}{3}$

$\arg(z-(-4+0i)) = \frac{\pi}{3}$ half line from $(-4,0)$ @ $\frac{\pi}{3}$



Min $|z|$ when z perp to locus

$\frac{|z|}{4} = \sin\frac{\pi}{3}$

$|z| = 4\frac{\sqrt{3}}{2} = 2\sqrt{3}$

(14) $|z+1+i| = 2|z+4-2i|$

$|(x+1)+i(y+1)| = 2|(x+4)+i(y-2)|$

$(x+1)^2+(y+1)^2 = 4[(x+4)^2+(y-2)^2]$

$x^2+2x+1+y^2+2y+1 = 4[x^2+8x+16+y^2-4y+4]$

$3x^2+30x+3y^2-18y = -78$

$x^2+10x+y^2-6y = -26$

$(x+5)^2+(y-3)^2 = -26+25+9$

$(x+5)^2+(y-3)^2 = 8$ Circle centre $(-5,3)$ radius $2\sqrt{2}$

(15) (a) $\arg z = \frac{\pi}{3}$ half line centre $(0,0)$ angle $\frac{\pi}{3}$

$\arg(x+iy) = \frac{\pi}{3}$

$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{3}$

$y = x\sqrt{3}$ — (1)

$\arg(z-4) = \frac{\pi}{2}$ ~~half~~

$\arg(z-(4+0i)) = \frac{\pi}{2}$ half line centre $(4,0)$ angle $\frac{\pi}{2}$

eqn $x=4$ — (2)

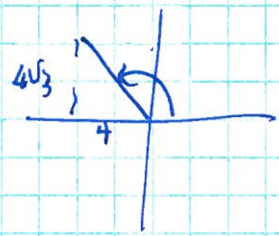
15 (a) cont'd $\arg(\cancel{(x-4)}+i$

a(i) $y = 4\sqrt{3}$

$\therefore \quad z = 4 + 4\sqrt{3}i$

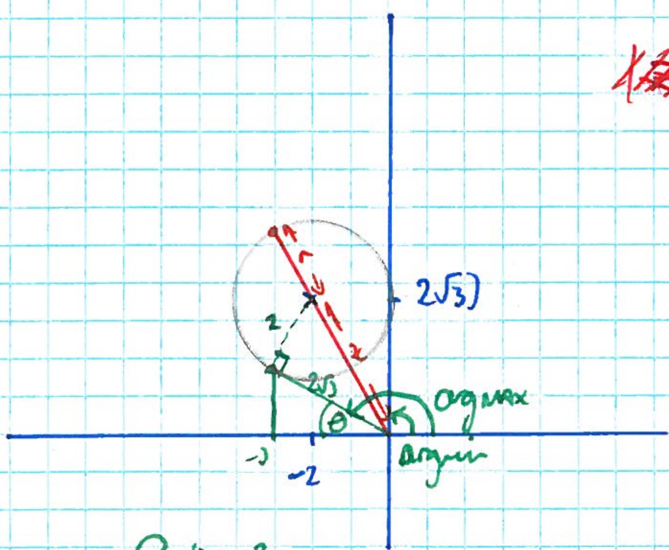
(b) Now $z - 8 = 4 + 4\sqrt{3}i - 8 = -4 + 4\sqrt{3}i$

$\arg(z-8) = \pi - \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \frac{2\pi}{3}$



16. $|z + 2 - 2\sqrt{3}i| = 2$

$|z - (-2 + 2\sqrt{3}i)| = 2$ circle centre $(-2, 2\sqrt{3})$ radius 2



$x^2 = 2^2 + (2\sqrt{3})^2$

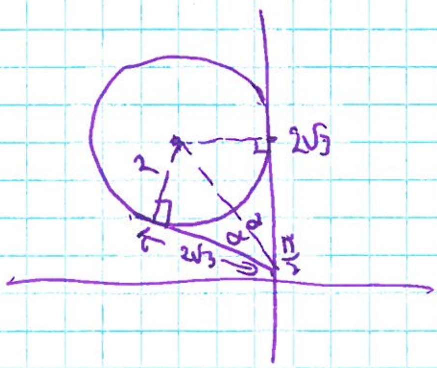
$x^2 = 4 + 12$

$x = 4$

$\cos\theta = \frac{3}{2\sqrt{3}}$

$\theta = \frac{\pi}{6}$ Arg max $= \frac{5\pi}{6}$, Arg min $= \frac{\pi}{2}$

Easier



$\tan\alpha = \frac{2}{2\sqrt{3}}$

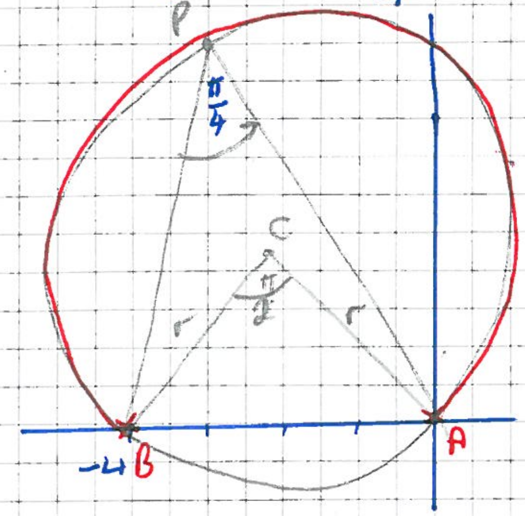
$\alpha = \frac{\pi}{6}$

$\arg_{max} = \frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{2} = \frac{5\pi}{6}$.

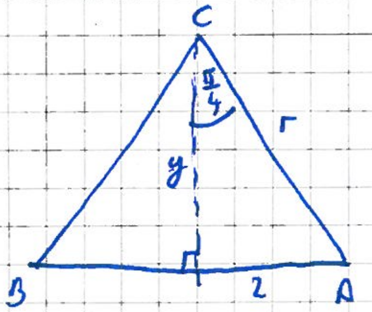
(17) $\arg z - \arg(z+4) = \frac{\pi}{4}$ arc of circle passing through $A(0,0)$ & $B(-4,0)$

$\vec{BP}$ to $\vec{AP}$ anti-clock

(a)



(b) Consider $\triangle ABC$



$\frac{2}{y} = \tan\frac{\pi}{4}$

$y = 2$

centre of circle @ $(-2,2)$

(c) $r^2 = 2^2 + 2^2$

$r^2 = 8$

$r = 2\sqrt{2}$

(d) $(x+2)^2 + (y-2)^2 = 8$

(e) Area of circle $= \pi \times 8 = 8\pi$

~~Area of segment below x-axis~~ $= \frac{\frac{\pi}{2}}{2\pi} \times 8\pi = 2\pi$

Area of sector CAB

Area of $\triangle ABC = \frac{1}{2} \times 8 \times \sin\frac{\pi}{2} = 4$

Area of segment below x-axis $= 2\pi - 4$

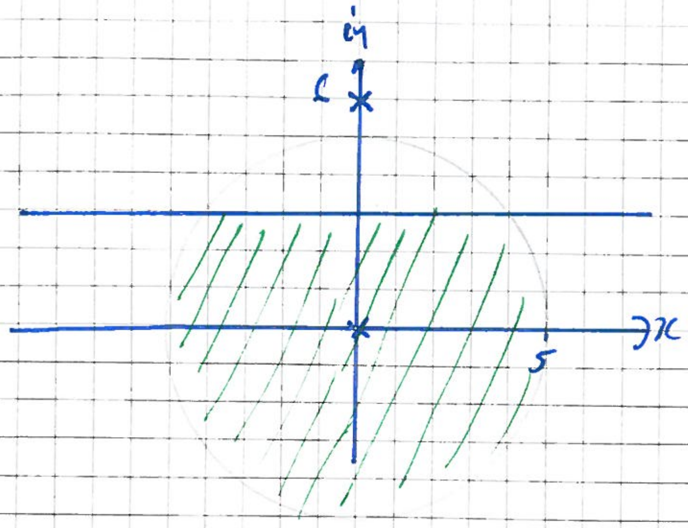
$\therefore$ Area bounded by locus & x-axis $= 8\pi - (2\pi - 4)$

$= \underline{6\pi + 4}$

Ex3G

(2) $|z| \leq 5$ circle centre $(0,0)$ radius 5

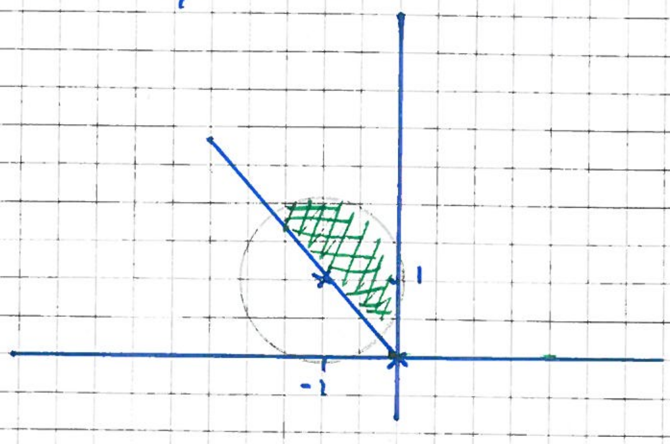
$|z| \leq |z-6i|$ perp bisector between $(0,0)$ and $(0,6)$



(3) $|z+1-i| \leq 1$

$|z-(-1+i)| \leq 1$ circle centre $(-1,1)$ radius 1

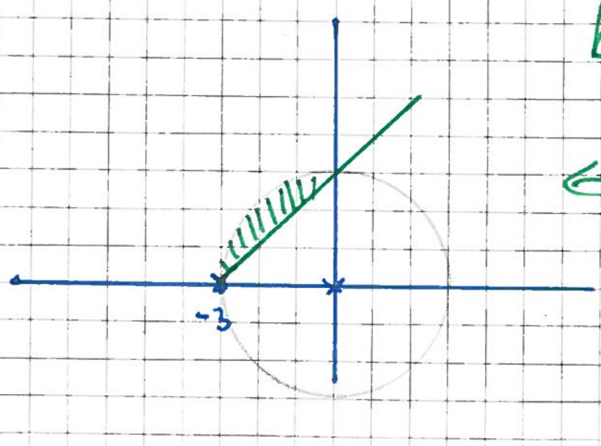
$0 \leq \arg z < \frac{3\pi}{4}$ half line from $(0,0)$ and angle between 0 & $\frac{3\pi}{4}$



(4) $|z| \leq 3$ circle centre $(0,0)$ radius 3.

$\frac{\pi}{4} \leq \arg(z+3) \leq \pi$ half line from $(-3,0)$ angle between $\frac{\pi}{4}$ and π

back of book suggests $\arg(z+3i)$.



← corrected.

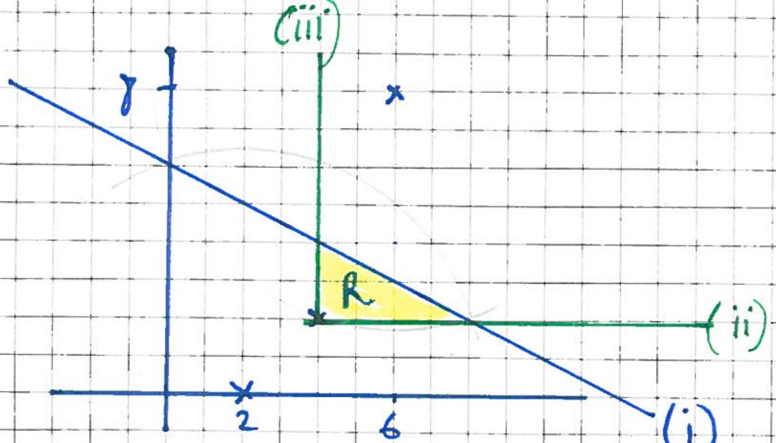
(5) (i) $|z-2| = |z-6-8i|$

$|z-(2)| = |z-(6+8i)|$ Perp bisect between $(2,0)$ & $(6,8)$

(ii) $\arg(z-(4+2i)) = 0$ half line from $(4,2)$ angle 0.

(iii) $\arg(z-(4+2i)) = \frac{\pi}{2}$ ... ... $\frac{\pi}{2}$

(b)



(6) (i) $|z+10| = |z-6-4\sqrt{2}i|$

$|z-(-10)| = |z-(6+4\sqrt{2}i)|$ perp bisect between $(-10,0)$ & $(6,4\sqrt{2})$

$|(x+10)+yi| = |(x-6)+(y-4\sqrt{2})i|$

$(x+10)^2+y^2 = (x-6)^2+(y-4\sqrt{2})^2$

$(x+10)^2-(x-6)^2 = (y-4\sqrt{2})^2-y^2$

$x^2+20x+100-(x^2-12x+36) = y^2-8\sqrt{2}y+32-y^2$

$32x+64 = 32-8\sqrt{2}y$

$8\sqrt{2}y = -32x-32$

$\sqrt{2}y = -4x-4$

$y = -\frac{4}{\sqrt{2}}(x+1)$

$y = -2\sqrt{2}(x+1)$ — (1)

(ii) $|z-(-1)| = 3$ circle centre $(-1,0)$ radius 3

$(x+1)^2+y^2 = 9$ — (2)

From (1) $(x+1) = -\frac{\sqrt{2}y}{4}$

in (2) $\left(-\frac{\sqrt{2}y}{4}\right)^2 + y^2 = 9$

(6) contd $\dfrac{4y^2}{16} + y^2 = 9$

$4y^2 + 16y^2 = 144$

$20y^2 = 144$

$y^2 = \dfrac{36}{5}$

$y = \pm \dfrac{6}{\sqrt{5}}$

(6) contd From (1) $(x+1) = -\dfrac{y}{2\sqrt{2}}$ — (3)

in (2) $\left(\dfrac{-y}{2\sqrt{2}}\right)^2 + y^2 = 9$

$\dfrac{y^2}{8} + y^2 = 9$

$9y^2 = 72$

$y^2 = 8$

$y = \pm 2\sqrt{2}$

in (3) $(x+1) = -\dfrac{(+2\sqrt{2})}{2\sqrt{2}} = -1$

$x = -2$ $\qquad z = -2 + 2\sqrt{2}i$

$(x+1) = -\dfrac{(-2\sqrt{2})}{2\sqrt{2}} = +1$

$x = 0$ $\qquad z = -2\sqrt{2}i$

(c)