Ex7F * Q1 the p.f. part has already been found Ex7E (i) → (iv) *

① (i)
$$\frac{4}{(1-3x)(1-x)^2} \equiv \frac{A}{(1-3x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

$$4 \equiv A(1-x)^2 + B(1-3x)(1-x) + C(1-3x)$$

$$\equiv A(1-2x+x^2) + B(1-4x+3x^2) + C(1-3x)$$

Comparing Coefficients

x^2: $A + 3B = 0$ — ①

x^1: $-2A - 4B - 3C = 0$ — ②

x^0: $A + B + C = 4$ — ③

from ① $A = -3B$ — ④

in ② $-2(-3B) - 4B - 3C = 0$

$$6B - 4B = 3C$$
$$2B = 3C$$
$$C = \frac{2B}{3} \quad — ⑤$$

in ③ $-3B + B + \frac{2}{3}B = 4$

$$B = -3$$

in ④ $A = +9$ in ⑤ $C = -2$

so

$$\frac{4}{(1-3x)(1-x)^2} \equiv \frac{9}{1-3x} - \frac{3}{1-x} - \frac{2}{(1-x)^2}$$

$$= 9(1-3x)^{-1} - 3(1-x)^{-1} - 2(1-x)^{-2}$$

(1)(i) $(1-3x)^{-1} = 1 + (-1)(-3x) + \frac{(-1)(-2)}{2!}(-3x)^2 + \dots$

$= 1 + 3x + 9x^2 + \dots$

$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \dots$

$= 1 + x + x^2 + \dots$

$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \dots$

$= 1 + 2x + 3x^2$

$\therefore \frac{4}{(1-3x)(1-x)^2} \approx 9[1+3x+9x^2] - 3[1+x+x^2] - 2[1+2x+3x^2]$

$\approx 9 + 27x + 81x^2 - 3 - 3x - 3x^2 - 2 - 4x - 6x^2$

$\approx 4 + 20x + 72x^2$

(ii) $\frac{4+2x}{(2x-1)(x^2+1)} \equiv \frac{A}{2x-1} + \frac{Bx+C}{x^2+1}$

$4 + 2x \equiv A(x^2+1) + (Bx+C)(2x-1)$

$\equiv Ax^2 + A + 2Bx^2 - Bx + 2Cx - C$

Compare Coeficients

x^2: $A + 2B = 0$ — (1)

x^1: $-B + 2C = 2$ — (2)

x^0: $A - C = 4$ — (3)

From (1) $A = -2B$ — (4)

From (2) ~~C = 2 + 2C~~ $C = \frac{2+B}{2}$ — (5)

In (3) $-2B - \left(\frac{2+B}{2}\right) = 4$

$\times 2$ $\quad -4B - 2 - B = 8$

$-5B = 10$

$B = -2$

In (4) $A = 4$

In (5) $C = 0$

$\therefore$ equiv $\frac{4}{2x-1} - \frac{2x}{x^2+1} = 4(2x-1)^{-1} - 2x(x^2+1)^{-1}$

$= 4\left[-1[1-2x]^{-1}\right] - 2x\left[(1+x^2)^{-1}\right]$

$(1-2x)^{-1} = 1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \ldots$

$= 1 + 2x + 4x^2$

$(1+x^2)^{-1} = 1 + (-1)(x^2) + \frac{(-1)(-2)(x^2)^2}{2!}$

$= 1 - x^2 + \cancel{x^4}$ only need up to quadratic term

$\therefore \frac{4}{2x-1} - \frac{2x}{x^2+1} \approx -4\left[1 + 2x + 4x^2\right] - 2x\left[1 - x^2\right]$

$\approx -4 - 8x - 16x^2 - 2x + \cancel{2x^3}$

$\approx -4 - 10x - 16x^2$

(iii) $$\frac{5-2x}{(x-1)^2(x+2)} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\equiv A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\equiv A[x^2+x-2] + B(x+2) + C(x^2-2x+1)$$

Compare Coeficients

x^2: $A + C = 0$ — (1)

x^1: $A + B - 2C = -2$ — (2)

x^0: $-2A + 2B + C = 5$ — (3)

from (1) $A = -C$ — (4)

in (2) $-C + B - 2C = -2$

$B = 3C - 2$ — (5)

in (3) $-2(-C) + 2(3C-2) + C = 5$

$2C + 6C - 4 + C = 5$

$9C = 9$

$C = 1$

$\therefore$ $A = -1$, $B = 1$

Hence $$\frac{5-2x}{(x-1)^2(x+2)} \equiv \frac{1}{(x-1)^2} - \frac{1}{(x-1)} + \frac{1}{x+2}$$

$$\equiv (x-1)^{-2} - (x-1)^{-1} + (x+2)^{-1}$$

$$\equiv [-1(1-x)]^{-2} - [-1(1-x)]^{-1} + \left[2\left[1+\frac{x}{2}\right]\right]^{-1}$$

$$\equiv -1^{-2}(1-x)^{-2} + (1-x)^{-1} + \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$\frac{+1}{(-1)^2} = +1$

(iii) contd

$$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \ldots$$

$$= 1 + 2x + 3x^2$$

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \ldots$$

$$= 1 + x + x^2$$

$$\left(1+\frac{x}{2}\right)^{-1} = 1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \ldots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{4} + \ldots$$

$$\therefore \frac{5-2x}{(x-1)^2(x+2)} \approx (1+2x+3x^2) + (1+x+x^2) + \frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}\right)$$

$$\approx \frac{5}{2} + \frac{11}{4}x + \frac{33}{8}x^2$$

(iv)
$$\frac{2x+1}{(x-2)(x^2+4)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$\equiv \frac{5}{8}(x-2)^{-1} + \frac{1}{8}(6-5x)(x^2+4)^{-1}$$

$$\equiv \frac{5}{8}\left[-2\left(1-\frac{x}{2}\right)\right]^{-1} + \frac{1}{8}(6-5x)\left[4\left(1+\frac{x^2}{4}\right)\right]^{-1}$$

$$= \frac{5}{8}\left[\frac{1}{-2}\left(1-\frac{x}{2}\right)^{-1}\right] + \frac{1}{8}(6-5x)\left[\frac{1}{4}\left(1+\frac{x^2}{4}\right)^{-1}\right]$$

$$= \frac{-5}{16}\left(1-\frac{x}{2}\right)^{-1} + \frac{1}{32}(6-5x)\left(1+\frac{x^2}{4}\right)^{-1}$$

(iv) Now $\left(1-\frac{x}{2}\right)^{-1} = 1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \dots$

$$= 1 + \frac{x}{2} + \frac{x^2}{4} + \dots$$

$$\left(1+\frac{x^2}{4}\right)^{-1} = 1 + (-1)\left(\frac{x^2}{4}\right) + \cancel{\frac{(-1)(-2)}{2!}\left(\frac{x^2}{4}\right)^2} + \dots$$

not required

$$= 1 - \frac{x^2}{4}$$

$$\cancel{\therefore \frac{2x+1}{(x-2)(x^2+4)} \approx -\frac{5}{16}\left[1+\frac{x}{2}+\frac{x^2}{4}\right] + (16-5x)\left[1-\frac{x^2}{4}\right]}$$

$$\cancel{\approx -\frac{5}{16} - \frac{5x}{32} - \frac{5x^2}{64} + 12 - 3x^2 - 10x + \frac{5x^3}{2}}$$

$\approx$

$$\therefore \frac{2x+1}{(x-2)(x^2+4)} \approx -\frac{5}{16}\left[1+\frac{x}{2}+\frac{x^2}{4}\right] + \frac{1}{32}(6-5x)\left[1-\frac{x^2}{4}\right]$$

$$\approx -\frac{5}{16} - \frac{5x}{32} - \frac{5x^2}{64} + \frac{1}{32}\left[6 - \frac{3x^2}{2} - 5x + \cancel{\frac{5x^3}{4}}\right]$$

$$\approx -\frac{1}{8} - \frac{5x}{16} - \frac{1}{8}x^2$$

2 (i) $\dfrac{7-4x}{(2x-1)(x+2)} \equiv \dfrac{A}{2x-1} + \dfrac{B}{x+2}$

$7-4x \equiv A(x+2) + B(2x-1)$

Compare Coefficients

$x^1: \quad A + 2B = -4 \quad — ①$

$x^0: \quad 2A - B = 7 \quad — ②$

from ① $\quad A = -2B - 4 \quad — ③$

in ② $\quad 2(-2B-4) - B = 7$

$-4B - 8 - B = 7$

$-5B = 15$

$B = -3$

in ③ $\quad A = -2(-3) - 4 = 2$

$\therefore \dfrac{7-4x}{(2x-1)(x+2)} = \dfrac{2}{2x-1} - \dfrac{3}{x+2}$

(ii) $\dfrac{1}{1-2x} = (1-2x)^{-1}$

$= 1 + (-1)(-2x) + \dfrac{(-1)(-2)(-2x)^2}{2!} + \dots$

$= 1 + 2x + 4x^2$

Valid for $|-2x| < 1$

$|x| < \frac{1}{2}$

(2) (iii) $\frac{1}{2+x} = (2+x)^{-1} = \left[2\left(1+\frac{x}{2}\right)\right]^{-1}$

$$= 2^{-1}\left(1+\frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \dots\right]$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$$

Valid for $\left|\frac{x}{2}\right| < 1$

$$|x| < 2$$

(iv) $\frac{7-4x}{(2x-1)(x+2)} \equiv 2(2x-1)^{-1} - 3(x+2)^{-1}$

$$\equiv 2\left[-1(1-2x)\right]^{-1} - 3(2+x)^{-1}$$

$$\equiv -2(1-2x)^{-1} - 3(2+x)^{-1}$$

$$\approx -2\left[1+2x+4x^2\right] - 3\left[\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}\right]$$

$$\approx -\frac{7}{2} - \frac{13x}{4} - \frac{67}{8}x^2$$

(v) when $x = 0.1$, $\frac{7-4(0.1)}{(2(0.1)-1)(0.1+2)} = -\frac{55}{14}$ $(\to A)$

$$-\frac{7}{2} - \frac{13(0.1)}{4} - \frac{67}{8}(0.1)^2 = -\frac{3127}{800} \quad (\to B)$$

% error $= \frac{B-A}{A} \times 100 = 0.505\ \%.$

③ (i) $(2-x)(1+x) = 2+x-x^2$

So $\dfrac{3x}{2+x-x^2} \equiv \dfrac{3x}{(2-x)(1+x)} \equiv \dfrac{A}{2-x} + \dfrac{B}{1+x}$

$3x \equiv A(1+x) + B(2-x)$

$x=-1$ $\quad -3 = 3B$

$B = -1$

$x = 2$ $\quad 6 = 3A$

$A = 2$

(ii) $\therefore \dfrac{3x}{2+x-x^2} \equiv \dfrac{2}{2-x} - \dfrac{1}{1+x}$

$\equiv 2\left[(2-x)^{-1}\right] - (1+x)^{-1}$

$\equiv 2\left[2\left(1-\frac{x}{2}\right)\right]^{-1} - (1+x)^{-1}$

$\equiv 2\left[\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}\right] - (1+x)^{-1}$

$\equiv \left(1-\frac{x}{2}\right)^{-1} - (1+x)^{-1}$

$\left(1-\frac{x}{2}\right)^{-1} = 1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \dots$

$= 1 + \frac{x}{2} + \frac{x^2}{4}$

Valid for $\left|-\frac{x}{2}\right| < 1$

$|x| < 2$

③(ii) contd

$$(1+x)^{-1} = 1 + (-1)(x) + \frac{(-1)(-2)}{2!}(x)^2 + \dots$$

$$= 1 - x + x^2$$

Valid for $|x|<1$

Hence $\frac{3x}{2+x-x^2} \simeq \left(1 + \frac{x}{2} + \frac{x^2}{4}\right) - \left(1 - x + x^2\right)$

$$\simeq \frac{3}{2}x - \frac{3}{4}x^2 + \dots$$

Valid for the strictest condition, ie $|x|<1$

④(i) $f(x) = \frac{8x-6}{(1-x)(3-x)} \equiv \frac{A}{1-x} + \frac{B}{3-x}$

So $8x - 6 \equiv A(3-x) + B(1-x)$

$x=3$ $\quad 18 = -2B$

$B = -9$

$x=1$ $\quad 2 = 2A$

$A = 1$

$\therefore f(x) = \frac{1}{1-x} - \frac{9}{3-x}$

$$= (1-x)^{-1} - 9(3-x)^{-1}$$

$$= (1-x)^{-1} - 9\left[3\left(1-\frac{x}{3}\right)\right]^{-1}$$

$$= (1-x)^{-1} - 9\left[3^{-1}\left(1-\frac{x}{3}\right)^{-1}\right]$$

(4)(i) contd $F(x) = (1-x)^{-1} - 9\left[\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}\right]$

$$F(x) = (1-x)^{-1} - 3\left(1-\frac{x}{3}\right)^{-1}$$

Now $F'(x) = -1(1-x)^{-2}.-1 - \cancel{3}.(-1)\left(1-\frac{x}{3}\right)^{-2}.\left(\frac{-1}{\cancel{3}}\right)$

$$= (1-x)^{-2} - \left(1-\frac{x}{3}\right)^{-2} \text{ As required.}$$

(ii) @ turning points, $F'(x) = 0$

$$\therefore \quad \frac{1}{(1-x)^2} - \left(\frac{3-x}{3}\right)^{-2} = 0$$

$$\frac{1}{(1-x)^2} = \left(\frac{3}{3-x}\right)^2$$

$$\frac{1}{(1-x)^2} = \frac{9}{(3-x)^2}$$

$$(3-x)^2 = 9(1-x)^2$$

$$9 - 6x + x^2 = 9(1-2x+x^2)$$

$$9 - 6x + x^2 = 9 - 18x + 9x^2$$

$$8x^2 - 12x = 0$$

$$4x(2x-3) = 0$$

$\therefore$ either $4x = 0 \Rightarrow \underline{x = 0}$

or $2x - 3 = 0 \Rightarrow \underline{x = 3/2}$

(4)(iii) $f'(x) = (1-x)^{-2} - \left(1 - \frac{x}{3}\right)^{-2}$

$$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \ldots$$

$$= 1 + 2x + 3x^2 + \ldots$$

$$\left(1 - \frac{x}{3}\right)^{-2} = 1 + (-2)\left(\frac{-x}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{-x}{3}\right)^2 + \ldots$$

$$= 1 + \frac{2x}{3} + \frac{x^2}{3} + \ldots$$

$$\therefore f'(x) = (1 + 2x + 3x^2) - \left(1 + \frac{2}{3}x + \frac{x^2}{3}\right)$$

$$\approx \cancel{\tfrac{8x}{3}}\ \frac{4x}{3} + \frac{8}{3}x^2$$

(iv) For both expansions, for even ~~pos~~ power of x, coeficient is +ve and x term is +ve

for odd powers of x, coefficient is −ve and x term is −ve

$\therefore$ always +ve evaluated coeficients.