Ex 7A

(1)(i)(a) $(1+x)^{-2}$, $n=-2$

$$(1+x)^{-2} = 1+(-2)x + \frac{(-2)(-3)}{2!}x^2 + \dots$$

$$= 1-2x+3x^2+\dots$$

(b) Valid for $|x|<1$

(c) when $x=0.1$ $(1+0.1)^{-2} = \frac{100}{121}$

$$1-2x+3x^2 = 0.83$$

relative error $= \frac{0.83 - 100/121}{100/121} \times 100 = 0.43\%$

(ii)(a) $\frac{1}{1+2x} = (1+2x)^{-1}$ let $x=(2x)$, $n=-1$

$$(1+2x)^{-1} = 1+(-1)(2x) + \frac{(-1)(-2)(2x)^2}{2!} + \dots$$

$$= 1-2x+4x^2$$

(b) Valid for $|2x|<1$

$$2|x|<1$$

$$|x|<\tfrac{1}{2}$$

(c) When $x=0.1$, $(1+2(0.1))^{-1} = 0.8\dot{3}$

$$1-2x+4x^2 = 0.84$$

relative error $= \frac{0.84-0.8\dot{3}}{0.8\dot{3}} \times 100 = 0.8\%$

1 (iii)(a) $\sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$, $x = (-x^2)$, $n = \frac{1}{2}$

$$(1-x^2)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)(-x^2) + \frac{\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)(-x^2)^2}{2!}$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4$$

(b) Valid for $|-x^2| < 1$

$|x| < 1$

(c) when $x = 0.1$, $\sqrt{1-(0.1)^2} = 0.994\ldots$

$1 - \frac{1}{2}(0.1)^2 - \frac{1}{8}(0.1)^4 = 0.9949875$

Relative error $= 6.3 \times 10^{-6}\,\%$.

(iv) $\frac{1+2x}{1-2x} = (1+2x)(1-2x)^{-1}$

find $(1-2x)^{-1}$ with $x = (-2x)$ $n = -1$

$$(1-2x)^{-1} = 1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2$$

$$= 1 + 2x + 4x^2$$

So $(1+2x)(1+2x+4x^2) = 1 + 2x + 4x^2 + 2x + 4x^2 + 8x^3$

$= 1 + 4x + 8x^2 + 8x^3$

So first 3 non zero terms $= 1 + 4x + 8x^2$

(b) Valid for $|-2x| < 1$

$2|x| < 1$

$|x| < \frac{1}{2}$

(c) Relative Error $= \frac{1.5 - 1.48}{1.5} \times 100$

$= 1.3\%$

(v)(a) $(3+x)^{-1} = \left[3\left(1+\frac{x}{3}\right)\right]^{-1} = 3^{-1}\left(1+\frac{x}{3}\right)^{-1} = \frac{1}{3}\left(1+\frac{x}{3}\right)^{-1}$

use $x = \left(\frac{x}{3}\right)$, $n = -1$

$$\frac{1}{3}\left(1+\frac{x}{3}\right)^{-1} = \frac{1}{3}\left[1 + (-1)\left(\frac{x}{3}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{3}\right)^2 + \dots\right]$$

$$= \frac{1}{3}\left[1 - \frac{x}{3} + \frac{2}{18}x^2\right]$$

$$= \frac{1}{3} - \frac{x}{9} + \frac{x^2}{27}$$

(b) Valid for $\left|\frac{x}{3}\right| < 1$

$$\frac{|x|}{3} < 1$$

$$|x| < 3$$

(c) Relative error $= 0.0037\%$

(vi)(a) $(1-x)\sqrt{4+x}$

$$(1-x)\left[4\left(1+\frac{x}{4}\right)\right]^{\frac{1}{2}}$$

$$2(1-x)\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

Use $x = \left(\frac{x}{4}\right)$ $n = \frac{1}{2}$

$$\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)^2 \times \frac{1}{2!}$$

$$= 1 + \frac{x}{8} - \frac{x^2}{128}$$

(vi) contd Now $2(1-x)\left[1+\frac{x}{8}-\frac{x^2}{128}\right]$

$= 2+\frac{x}{4}-\frac{x^2}{64}-2x-\frac{x^2}{4}+\frac{x^3}{64}$

$= 2-\frac{7x}{4}-\frac{17x^2}{64}$

(b) Valid for $\left|\frac{x}{4}\right|<1$

$|x|<4$

(c) Relative error $= 9.5\times10^{-4}\,\%$

(vii) $\frac{x+2}{x-3} = (x+2)(x-3)^{-1} =$

$= (x+2)\left[-3+x\right]^{-1}$

$= (x+2)\left[-3\left(1+\frac{x}{-3}\right)\right]^{-1} = -\frac{1}{3}(x+2)\left(1-\frac{x}{3}\right)^{-1}$

let $x=\left(-\frac{x}{3}\right), n=-1$

$\left(1-\frac{x}{3}\right)^{-1} = 1+(-1)\left(-\frac{x}{3}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{3}\right)^2$

$= 1+\frac{x}{3}+\frac{x^2}{9}$

$\therefore\ -\frac{1}{3}(x+2)\left(1+\frac{x}{3}+\frac{x^2}{9}\right)$

$= -\frac{x}{3}-\frac{x^2}{9}-\frac{x^3}{27}-\frac{2}{3}-\frac{2x}{9}-\frac{2x^2}{27} = -\frac{2}{3}-\frac{5x}{9}-\frac{5x^2}{27}$

(vii) (b) Valid for $\left|-\frac{x}{3}\right| < 1$

$|x| < 3$

(c) Relative error $= 8.8 \times 10^{-3}$

(viii) $\frac{1}{\sqrt{3x+4}} = (4+3x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left(1+\frac{3x}{4}\right)^{-\frac{1}{2}}$

let $x = \frac{3x}{4}$, $n = -\frac{1}{2}$

So $\frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{3x}{4}\right) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{3x}{4}\right)^2 \times \frac{1}{2!}\right]$

$\frac{1}{2}\left[1 - \frac{3x}{8} + \frac{27}{128}x^2\right]$

$= \frac{1}{2} - \frac{3x}{16} + \frac{27}{256}x^2$

Valid for $\left|\frac{3x}{4}\right| < 1$

$|x| < \frac{4}{3}$

Relative Error $= 0.013\%$.

(ix) $\dfrac{1+2x}{(2x-1)^2} = (1+2x)\left[-1+2x\right]^{-2} = (1+2x)\left[-1(1-2x)\right]^{-2}$

$= (1+2x)(1-2x)^{-2}$

let $x = (-2x)$ $n = -2$

$(1+2x)\left[1 + (-2)(-2x) + \dfrac{(-2)(-3)}{2!}(-2x)^2\right]$

$= (1+2x)\left[1 + 4x + 12x^2\right]$

$= 1 + 4x + 12x^2 + 2x + 8x^2 + 24x^3$

$= 1 + 6x + 20x^2$

Valid for $|2x| < 1$

$|x| < \dfrac{1}{2}$

Relative error 4%.

(x) $\dfrac{1+x^2}{1-x^2} = (1+x^2)(1-x^2)^{-1}$

let $x = (-x^2)$, $n = -1$

$(1+x^2)\left[1 + (-1)(-x^2) + \dfrac{(-1)(-2)}{2!}(-x^2)^2\right]$

$= (1+x^2)\left[1 + x^2 + \cancel{2}x^4\right]$

$= 1 + x^2 + \cancel{2}x^4 + x^2 + x^4 + \cancel{2}x^6$

$= 1 + 2x^2 + 2x^4$ Valid for $|-x^2| < 1$

$|x| < 1$

(xi) $\sqrt[3]{1+2x^2} = (1+2x^2)^{1/3}$ let $x = (2x^2)$ $n = \frac{1}{3}$

$= 1 + \left(\frac{1}{3}\right)(2x^2) + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(2x^2)^2}{2!}$

$= 1 + \frac{2}{3}x^2 - \frac{8}{18}x^4$

$= 1 + \frac{2}{3}x^2 - \frac{4}{9}x^4$

Valid for $|2x^2| < 1$

$|x^2| < \frac{1}{2}$

$|x| < \frac{1}{\sqrt{2}}$

(xii) $\frac{1}{(1+2x)(1+x)} = (1+2x)^{-1}(1+x)^{-1}$

$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)(2x)^2}{2!} = 1 - 2x + 4x^2$

Valid for $|2x| < 1$

$|x| < \frac{1}{2}$

$(1+x)^{-1} = 1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} = 1 - x + x^2$

Valid for $|x| < 1$

both satisfied for $|x| < \frac{1}{2}$

Now $(1-2x+4x^2)(1-x+x^2)$

$= 1 - x + x^2 - 2x + 2x^2 - 2x^3 + 4x^2 - 4x^3 + 4x^4$

$= 1 - 3x + 7x^2$

② (i)

$$\begin{array}{c} 1 \\ 1\quad 1 \\ 1\quad 2\quad 1 \\ 1\quad 3\quad 3\quad 1 \end{array}$$

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$

(ii) $(1-x)^{-4} = 1 + (-4)(-x) + \frac{(-4)(-5)}{2!}(-x)^2 + \frac{(-4)(-5)(-6)}{3!}(-x)^3 + \ldots$

$= 1 + 4x + 10x^2 + 20x^3$

Valid for $|-x|<1$

$|x|<1$

(iii) $\frac{(1+x)^3}{(1-x)^4} = (1+x)^3(1-x)^4$

$= (1+3x+3x^2+x^3)(1+4x+10x^2+20x^3)$

$= (1+4x+10x^2+20x^3)+(3x+12x^2+30x^3+\ldots)+(3x^2+12x^3+\ldots)+(x^3+\ldots)$

$= 1 + 7x + 25x^2 + 63x^3$

$\therefore\ a=25\ ,\ b=63$

(3) (i)

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$(2-x)^4 = (1)(2)^4(-x)^0 + 4(2)^3(-x)^1 + 6(2)^2(-x)^2 + 4(2)^1(-x)^3 + (1)(2)^0(-x)^4$$

$$= 16 - 32x + 24x^2 - 8x + x^4$$

(ii) $$(1+2x)^{-3} = 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(2x)^3$$

$$= 1 - 6x + 24x^2 - 80x^3$$

Valid for $|2x| < 1$

$|x| < \frac{1}{2}$

(iii) $$\frac{(2-x)^4}{(1+2x)^3} = (2-x)^4(1+2x)^{-3}$$

$$= (16 - 32x + 24x^2 - 8x^3 + x^4)(1 - 6x + 24x^2 - 80x^3)$$

Need quadratic terms

$$= (16 - 96x + 384x^2 \ldots +)(-32x + 192x^2 + \ldots) + 24x^2$$

$$= 16 - 128x + 600x^2$$

$\therefore$ $a = -128$, $b = 600$

4 (i) $(1-x)^{-1}$, $x = (-x)$ $n = -1$

$$= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \ldots$$

$$= 1 + x + x^2 + x^3$$

Valid $|-x| < 1$

$|x| < 1$

(ii) $(1+2x)^{-2}$, $x = (2x)$ $n = -2$

$$= 1 + (-2)(2x) + \frac{(-2)(-3)}{2!}(2x)^2 + \frac{(-2)(-3)(-4)}{3!}(2x)^3$$

$$= 1 - 4x + \frac{24}{2}x^2 - \frac{192}{6}x^3$$

$$= 1 - 4x + 12x^2 - 32x^3$$

Valid for $|2x| < 1$

$|x| < \frac{1}{2}$

(iii) $\frac{1}{(1-x)(1+2x)^2} = (1-x)^{-1}(1+2x)^{-2}$

$$(1 + x + x^2 + x^3)(1 - 4x + 12x^2 - 32x^3)$$

$$= (1 - 4x + 12x^2 - 32x^3) + (x - 4x^2 + 12x^3) + (x^2 - 4x^3 + \ldots) + x^3$$

$$= 1 - 3x + 9x^2 - 23x^3$$

Valid for $|x| < \frac{1}{2}$

(5) (i) $\dfrac{1}{\sqrt{4-x}} = (4-x)^{-1/2} = \left[4\left(1-\dfrac{x}{4}\right)\right]^{-\frac{1}{2}}$

$= 4^{-1/2}\left(1-\dfrac{x}{4}\right)^{-1/2}$

$= \dfrac{1}{\sqrt{4}}\left(1-\dfrac{x}{4}\right)^{-1/2}$

$= \dfrac{1}{2}\left(1-\dfrac{x}{4}\right)^{-1/2}$ As required.

(ii) $\left(1-\dfrac{x}{4}\right)^{-1/2} = 1 + \left(-\dfrac{1}{2}\right)\left(-\dfrac{x}{4}\right) + \left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\left(-\dfrac{x}{4}\right)^2 \times \dfrac{1}{2!}$

$= 1 + \dfrac{x}{8} + \dfrac{3}{128}x^2$

Valid for $\left|\dfrac{-x}{4}\right| < 1$

$|x| < 4$

(iii) $\dfrac{2(1+x)}{\sqrt{4-x}} = \cancel{2}(1+x) \cdot \dfrac{1}{\cancel{2}}\left(1-\dfrac{x}{4}\right)^{-1/2}$

$= (1+x)\left[1 + \dfrac{x}{8} + \dfrac{3}{128}x^2\right]$

$= 1 + \dfrac{x}{8} + \dfrac{3}{128}x^2 + x + \dfrac{x^2}{8} + \dfrac{3x^3}{128}$

$= 1 + \dfrac{9}{8}x + \dfrac{19}{128}x^2$

6 (i) $(1+y)^{-1} = 1 + (-1)(y) + \frac{(-1)(-2)}{2!}(y)^2 + \frac{(-1)(-2)(-3)}{3!}(y)^3$

$= 1 - y + y^2 - y^3$

(ii) $\left(1+\frac{2}{x}\right)^{-1}$ let $y = \left(\frac{2}{x}\right)$

$= 1 - \left(\frac{2}{x}\right) + \left(\frac{2}{x}\right)^2 - \left(\frac{2}{x}\right)^3$

$= 1 - \frac{2}{x} + \frac{4}{x^2} - \frac{8}{x^3}$

(iii) $\left(1+\frac{2}{x}\right)^{-1} = \left(\frac{x+2}{x}\right)^{-1} = \frac{x}{x+2}$ As required

$\left(1+\frac{2}{x}\right)^{-1} = \left[\frac{2}{x}\left(\frac{x}{2}+1\right)\right]^{-1}$

$= \left(\frac{2}{x}\right)^{-1}\left(\frac{x}{2}+1\right)^{-1}$

$= \frac{x}{2}\left(1+\frac{x}{2}\right)^{-1}$ As required.

(iv) $\frac{x}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{x}{2}\left[1 - \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3\right]$

$= \frac{x}{2}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}\right]$

$= \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} - \frac{x^4}{16}$

(v) For $\left(1+\frac{2}{x}\right)^{-1}$, $\left|\frac{2}{x}\right| < 1 \Rightarrow \frac{2}{|x|} < 1 \Rightarrow 2 < |x|$

ie $x < -2$ or $x > 2$

For $\frac{x}{2}\left(1+\frac{x}{2}\right)^{-1}$, $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2 \Rightarrow -2 < x < 2$

one has x as numerator, the other as denominator

7 (i) $\int_1^k t\ln t \, dt$ $u = \ln t$ $\frac{dv}{dt} = t$

$\frac{du}{dt} = \frac{1}{t}$ $v = \frac{t^2}{2}$

$$I = \left[\frac{t^2}{2}\ln t\right]_1^k - \int_1^k \frac{1}{t}\cdot\frac{t^2}{2}\, dt$$

$$= \frac{k^2}{2}\ln k - \frac{1}{2}\int_1^k t \, dt$$

$$= \frac{k^2}{2}\ln k - \frac{1}{2}\left[\frac{t^2}{2}\right]_1^k$$

$$= \frac{k^2}{2}\ln k - \frac{1}{4}\left(k^2 - 1\right)$$

$$= \frac{1}{2}k^2\ln k - \frac{1}{4}k^2 + \frac{1}{4}$$ As required

(ii) $$(1-2x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-2x) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^2 \times \frac{1}{2!}$$

$$+ \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-2x)^3 \times \frac{1}{3!}$$

$$= 1 + x + \frac{12}{8}x^2 + \frac{120}{48}x^3$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$$

Valid for $|-2x| < 1$

$2|x| < 1$

$|x| < \frac{1}{2}$

7(iii) $(1-2x)^{-\frac{1}{2}}\ln(1+x) \simeq \left(1+x+\frac{3}{2}x^2+\frac{5}{2}x^3\right)\left(\ln(1+x)\right)$

For small values of x, $\ln(1+x)$ is small as are multiples with x^2+x^3

$\therefore$ $(1-2x)^{-\frac{1}{2}}\ln(1+x) \approx (1+x)\ln(1+x)$

$\simeq t\ln t$, where $t=1+x$

$$\int_0^{0.1} \frac{\ln(1+x)}{\sqrt{1-2x}}\,dx \simeq \int_0^{0.1} (1-2x)^{-\frac{1}{2}}\ln(1+x)\,dx$$

For $x=0.1$, $t=1.1$
$x=0$, $t=1$

$t=1+x$

$\frac{dt}{dx}=1$, $dx=dt$

$\therefore$ $\int_0^{0.1}(1-2x)^{-\frac{1}{2}}\ln(1+x)\,dx \approx \int_1^{1.1} t\ln t\,dt$

From (i) $I = \frac{1}{2}(1.1)^2\ln(1.1) - \frac{1}{4}(1.1)^2 + \frac{1}{4}$

$= 0.00516$ to 5dp.