Ex 3E   Ex 10B

1 (a) $\int \frac{1}{(1-x)(3x-2)} \, dx$

$\frac{A}{1-x} + \frac{B}{3x-2} = \frac{A(3x-2)+B(1-x)}{\text{LCM}}$

Now $3A + B = 0$ — ①

$-2A + B = 1$ — ②

① + ② $A = 1$ in ① $B = -3$

$\therefore \int \frac{1}{1-x} + -\frac{3}{3x-2} \, dx$

$\int \frac{1}{1-x} \, dx - \int \frac{3}{3x-2} \, dx$

let $u = 1-x$ $\frac{du}{dx} = -1$ $dx = \frac{du}{-1}$

$\int \frac{1}{u} \cdot \frac{du}{-1} - \int \frac{\cancel{3}}{v} \frac{dv}{\cancel{3}}$

$-\ln|1-x| - \ln|3x-2| + c$

$3A - B = 0$ — ①

$-2A + B = 1$ — ②

① + ② $A = 1$ in ① $B = 3$

$\therefore \int \frac{1}{1-x} + \frac{3}{3x-2} \, dx$

$\int \frac{1}{1-x} \, dx + \int \frac{3}{3x-2} \, dx$

let $v = 3x-2$ $\frac{dv}{dx} = 3$ $dx = \frac{dv}{3}$

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let $u = 1-x$ $\frac{du}{dx} = -1$ $dx = -du$ let $v = 3x-2$ $\frac{dv}{dx} = 3$ $dx = \frac{dv}{3}$

$\int \frac{1}{u} \cdot \frac{du}{-1} + \int \frac{\cancel{3}}{v} \cdot \frac{dv}{\cancel{3}}$

$= -\ln|1-x| + \ln|3x-2| + c$

$= \ln\left|\frac{3x-2}{1-x}\right| + c$

(b) $\int \frac{7x-2}{(x-1)^2(2x+3)}\,dx \equiv \int \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}\,dx$

$= A(2x+3) + \underset{\text{LCM}}{B(x-1)(2x+3)} + C(x-1)^2$

$= A(2x+3) + \underset{\text{LCM}}{B(2x^2+x-3)} + C(x^2-2x+1)$

$B + C = 0$ — ①

$2A + B - 2C = 7$ — ②

$3A - 3B + C = -2$ — ③

from ① $B = -C$

in ② $2A - C - 2C = 7$

$2A - 3C = 7$

$A = \frac{7+3C}{2}$

in ③ $3\left(\frac{7+3C}{2}\right) - 3(-C) + C = -2$

$\times 2$ $\quad 21 + 9C + 6C + 2C = -4$

$17C = -25$

$C = \frac{-25}{17}$

$B = \frac{25}{17}$ $\quad A = \frac{7 - \frac{75}{17}}{2} = \frac{22}{17}$

$2B + C = 0$ — ①

$2A + B - 2C = 7$ — ②

$3A - 3B + C = -2$ — ③

From ① $C = -2B$

in ② $2A + B - 2(-2B) = 7$

$2A + B + 4B = 7$

$2A + 5B = 7$

$A = \frac{7-5B}{2}$

in ③ $3\left(\frac{7-5B}{2}\right) - 3B - 2B = -2$

$\times 2$ $\quad 21 - 15B - 6B - 4B = -4$

$21 - 25B = -4$

$B = 1.$

$\therefore C = -2$

$A = 1.$

$$\int \frac{1}{(x-1)^2} + \frac{1}{(x-1)} - \frac{2}{(2x+3)}\,dx$$

let $u_1 = x-1$ $\quad \frac{du_1}{dx} = 1$ $\quad dx = du$

let $u_2 = x-1$ $\quad \frac{du}{dx} = 1$ $\quad dx = du$

let $u_3 = 2x+3$ $\quad \frac{du}{dx} = 2$ $\quad dx = \frac{du}{2}$

$$\int u^{-2} + \frac{1}{u} - \frac{\cancel{2}1}{u} \cdot \frac{du}{\cancel{2}}$$

$$= \frac{u^{-1}}{-1} + \ln u - \ln u$$

$$= -\frac{1}{(x-1)} + \ln|x-1| - \ln|2x+3| + c = \ln\left|\frac{x-1}{2x+3}\right| - \frac{1}{(x-1)} + c$$

① (b) $\displaystyle\int \frac{7x-2}{(x-1)^2(2x+3)}\,dx \equiv \int \frac{A}{(x-1)^2} + \frac{B}{\quad}$ done.

(c) $\displaystyle\int \frac{x+1}{(x^2+1)(x-1)}\,dx \equiv \int \frac{Ax+B}{x^2+1} + \frac{C}{x-1}\,dx$

$\displaystyle\equiv \int \frac{(Ax+B)(x-1) + C(x^2+1)}{LCM}\,dx$

$\displaystyle\equiv \int \frac{Ax^2 - Ax + Bx - B + Cx^2 + C}{LCM}\,dx$

$\displaystyle= \int \frac{x^2(A+C) + x(B-A) - B + C}{LCM}\,dx$

x^2: $A+C=0$ — ①

x: $B-A=1$ — ②

$-B+C=1$ — ③

from ② $A=B-1$ — ④

in ① $\cancel{B+C=0}$ $B-1+C=0$

$\cancel{B=-C}$ $B=1-C$ — ⑤

in ③ $-(1-C)+C)=1$

$-1+C+C=1$

$2C=2$

$C=1$

in ⑤ $B=0$ in ④ $A=-1$

$\therefore \displaystyle\int \frac{x+1}{(x^2+1)(x-1)}\,dx \equiv \int \frac{-x}{(x^2+1)}\,dx + \int \frac{1}{(x-1)}\,dx$

let $u_1 = x^2+1$ $\frac{du_1}{dx} = 2x$ $dx = \frac{du_1}{2x}$ let $u_2 = x-1$ $\frac{du_2}{dx} = 1$ $du_2 = dx$

$\therefore \displaystyle\int -\frac{x}{u_1}\cdot\frac{du_1}{2x} + \int \frac{1}{u}\,du = -\frac{1}{2}\ln|x^2+1| + \ln|x-1| + c$

$= \ln|x-1| - \ln\left|\sqrt{x^2+1}\right| = \ln\left|\frac{x-1}{\sqrt{x^2+1}}\right| + c$

(1)(e) $\int \frac{1}{x^2(1-x)} dx \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x}$

$$\equiv \frac{A(1-x) + Bx(1-x) + Cx^2}{\text{LCM}}$$

$$\equiv \frac{A - Ax + Bx - Bx^2 + Cx^2}{\text{LCM}}$$

x^2: $C - B = 0$ — (1)

x: $B - A = 0$ — (2)

$A = 1$

in (2) $B - 1 = 0 \quad \therefore B = 1$

in (1) $C - 1 = 0 \quad \therefore C = 1$

$\therefore \int \frac{1}{x^2(1-x)} dx \equiv \int \frac{1}{x^2} dx + \int \frac{1}{x} dx + \int \frac{1}{1-x} dx \qquad u = 1-x \quad \frac{du}{dx} = -1$

~~let $u = x^2$, $\frac{du}{dx} = 2x$, $dx = du$~~

$$\equiv \int x^{-2} dx + \int \frac{1}{x} dx + \int \frac{1}{u} \frac{du}{-1}$$

$$\equiv \frac{x^{-1}}{-1} + \ln|x| - \ln|1-x| + c$$

$$\equiv -\frac{1}{x} + \ln\left|\frac{x}{1-x}\right| + c$$

$$\equiv \ln\left|\frac{x}{1-x}\right| - \frac{1}{x} + c$$

① (a) $\int \frac{2x-4}{(x^2+4)(x+2)} dx \equiv \int \frac{Ax+B}{x^2+4} + \frac{C}{x+2} dx$

$\equiv \frac{(Ax+B)(x+2) + C(x^2+4)}{\text{Lcm}}$

$\equiv \frac{Ax^2 + 2Ax + Bx + 2B + Cx^2 + 4C}{\text{Lcm}}$

x^2: $A + C = 0$ — ①

x: $2A + B = 2$ — ②

$2B + 4C = -4$ — ③

from ① $A = -C$ — ④

in ② $2(-C) + B = 2$

$-2C + B = 2$

$B = 2 + 2C$ — ⑤

in ③ $2(2+2C) + 4C = -4$

$4 + 4C + 4C = -4$

$8C = -8$

$C = -1$

in ④ $A = 1$ in ⑤ $B = 2 - 2 = 0.$

$\therefore \int \frac{x}{x^2+4} dx - \int \frac{1}{x+2} dx$

let $u_1 = x^2 + 4$

$\frac{du_1}{dx} = 2x$

$dx = \frac{du_1}{2x}$

let $u_2 = x + 2$

$\frac{du_2}{dx} = 1$

$du_2 = dx$

$\therefore \int \frac{x}{u_1} \cdot \frac{du_1}{2x} - \int \frac{1}{u_2} \cdot du_2$

$= \frac{1}{2} \ln|x^2+4| - \ln|x+2| + C$

$= \ln\left|\frac{\sqrt{x^2+4}}{x+2}\right| + C$

(2) $f(x) = \dfrac{3x+4}{(x^2+4)(x-3)} \equiv \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x-3}$ 1 mark correct form

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$\equiv \dfrac{(Ax+B)(x-3)+C(x^2+4)}{\text{lcm}}$

$\equiv \dfrac{Ax^2+Bx-3Ax-3B+Cx^2+4C}{\text{lcm}}$

x^2: $A+C=0$ — ①
x: $B-3A=3$ — ②
$4C-3B=4$ — ③

1 mark eqs.

From ① $A=-C$ — ④

in ② $B-3(-C)=3$

$B=3-3C$ — ⑤

in ③ $4C-3(3-3C)=4$

$4C-9+9C=4$

$13C=13$

$C=1$ 1

in ④ $A=-1$ 1 in ⑤ $B=0$ 1

$\therefore f(x) = \dfrac{1}{x-3} - \dfrac{x}{x^2+4}$ 1

Now $\displaystyle\int_0^2 \frac{1}{x-3}\,dx - \int_0^2 \frac{x}{x^2+4}\,dx$

let $u_1 = x-3$, $\dfrac{du_1}{dx}=1$, $dx = du_1$ 1

$u_2 = x^2+4$, $\dfrac{du_2}{dx} = 2x$, $dx = \dfrac{du_2}{2x}$ 1

when $x=0$ $u_1=-3$; $x=2$ $u_1 = -1$

when $x=0$ $u_2=4$; $x=2$ $u_2=8$

limits 1

$\therefore \displaystyle\int_{-3}^{-1} \frac{1}{u_1}\,du_1 - \int_4^8 \frac{x}{u_2}\frac{du_2}{2x}$ correct substs 1

$= \Big[\ln|u_1|\Big]_{-3}^{-1} - \Big[\tfrac{1}{2}\ln|u_2|\Big]_4^8$ correct int 1

$= \ln 1 - \ln 3 - \tfrac{1}{2}(\ln 8 - \ln 4)$

$= \ln 1 - \ln 3 - \tfrac{1}{2}\ln 8 + \tfrac{1}{2}\ln 4$

$= -1.445$ AND 1

$= \ln\left|\dfrac{1\times\sqrt{4}}{3\times\sqrt{8}}\right| = \ln\left|\dfrac{2}{6\sqrt{2}}\right|$

$= \ln\left|\dfrac{\cancel{2}}{\cancel{6}3\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}}\right|$

$= \ln\left|\dfrac{\sqrt{2}}{6}\right|$

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3. $\frac{1}{x^2(2x+1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{2x+1}$ 1 mark correct form

$\equiv \frac{A(2x+1) + Bx(2x+1) + Cx^2}{\text{LCM}}$

$\equiv \frac{2Ax + A + 2Bx^2 + Bx + Cx^2}{\text{LCM}}$

x^2: $2B + C = 0$ — ①

x: $2A + B = 0$ — ②

$A = 1$ 1 — ③

1 mark eq's

in ② $2 + B = 0$, $B = -2$ 1 in ① $2(-2) + C = 0$, $C = 4$ 1

$f(x) \equiv \frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1}$

$\int_1^2 \frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1}\, dx$ 1

let $u = 2x+1$ $\frac{du}{dx} = 2$ $dx = \frac{du}{2}$ 1

when $x = 1$ $u = 3$
$x = 2$ $u = 5$ 1 limits

$\int_1^2 x^{-2}\, dx - 2\int_1^2 \frac{1}{x}\, dx + 2\int_3^5 \frac{1}{u}\, du$ 1 substs.

$= \left[\frac{x^{-1}}{-1}\right]_1^2 - 2\left[\ln|x|\right]_1^2 + 2\left[\ln|u|\right]_3^5$ 1 mark int of x^{-1} 1 mark ln ints.

$= \left[-\frac{1}{2} + 1\right] - 2\left[\ln 2 - \ln 1\right] + 2\left[\ln 5 - \ln 3\right]$ 1 mark subst limits

$= \frac{1}{2} - 2\ln 2 + 2\ln 1 + 2\ln 5 - 2\ln 3$

$= \frac{1}{2} + \ln \frac{1^2 \times 5^2}{2^2 \times \cancel{1} \times 3^2} = \frac{1}{2} + \ln \frac{25}{36} = \frac{1}{2} + \ln\left(\frac{5}{6}\right)^2$. 1 mark proof

$= \frac{1}{2} + 2\ln\left(\frac{5}{6}\right)$

13.

(4) (a) (i) $\dfrac{3}{(1+x)(1-2x)} = \dfrac{A}{1+x} + \dfrac{B}{1-2x}$

$= \dfrac{A(1-2x) + B(1+x)}{\text{lcm}}$

$\therefore \quad A+B = 3 \quad -①$

$-2A+B = 0 \quad -②$

$① - ② \quad 3A = 3 \quad A = 1 \quad B = 2$

$\therefore \quad \dfrac{3}{(1+x)(1-2x)} = \dfrac{1}{1+x} + \dfrac{2}{1-2x}$

(ii) $\displaystyle\int_0^{0.1} \frac{1}{1+x}\,dx + \int_0^{0.1} \frac{2}{1-2x}\,dx$

let $u_1 = 1+x \quad \dfrac{du_1}{dx} = 1 \quad dx = du_1$

when $x=0 \quad u_1 = 1$

$x = 0.1 \quad u_1 = 1.1$

let $u_2 = 1-2x \quad \dfrac{du_2}{dx} = -2 \quad dx = \dfrac{du_2}{-2}$

when $x=0 \quad u_2 = 1$

$x = 0.1 \quad u_2 = 0.8$

$\therefore \quad \displaystyle\int_1^{1.1} \frac{1}{u_1}\,du_1 + \int_1^{0.8} \frac{2}{u_2}\cdot\frac{du_2}{-2} = \Big[\ln|u_1|\Big]_1^{1.1} - \Big[\ln|u_2|\Big]_1^{0.8}$

$= \ln 1.1 - \ln 1 - \ln 0.8 + \ln 1 = 0.31845$

(b) (i) $3(1+x)^{-1}(1-2x)^{-1}$

$(1+x)^{-1} = 1 + (-1)(x) + \dfrac{(-1)(-2)(x)^2}{2!} + \dots = 1 - x + x^2$

$(1-2x)^{-1} = 1 + (-1)(-2x) + \dfrac{(-1)(-2)(-2x)^2}{2!} + \dots = 1 + 2x + 4x^2$

$\therefore \quad 3(1-x+x^2)(1+2x+4x^2) = 3\left[1 + 2x + 4x^2 - x - 2x^2 - \cancel{4x^3} + x^2 + \dots\right]$

$= 3\left[1 + x + 3x^2\right] = 3 + 3x + 9x^2$

(ii) $\displaystyle\int_0^{0.1} 3 + 3x + 9x^2\,dx = \left[3x + \frac{3x^2}{2} + 3x^3\right]_0^{0.1} = 3(0.1) + \frac{3}{2}(0.1)^2 + 3(0.1)^3$

$= 0.31800$

(5) i) $\frac{x^2 - x - 24}{(x+2)(x-4)} \equiv \frac{A}{1} + \frac{B}{x+2} + \frac{C}{x-4}$

$x^2 - x - 24 \equiv A(x+2)(x-4) + B(x-4) + C(x+2)$

$x = -2$

$(-2)^2 - (-2) - 24 = 0 - 6B + 0$

$4 + 2 - 24 = -6B$

$-18 = -6B$

$B = 3$

$x = 4$ $\quad (4)^2 - (4) - 24 = 0 + 0 + 6C$

$16 - 4 - 24 = 6C$

$-12 = 6C$

$C = -2$

$x = 0$

$-24 = A(2)(-4) + 3(0-4) - 2(2)$

$-24 = -8A - 12 - 4$

$-8 = -8A$

$A = 1$

(ii) $\therefore \quad I = \int_1^3 1 + \frac{3}{x+2} - \frac{2}{x-4} \, dx$

$= \left[x + 3\ln|x+2| - 2\ln|x-4| \right]_1^3$

$= \left[x + \ln\left| \frac{(x+2)^3}{(x-4)^2} \right| \right]_1^3$

$= 3 + \ln\left(\frac{5^3}{(-1)^2}\right) - 1 - \ln\left(\frac{3^3}{(-3)^2}\right)$

$= 2 + \ln(125) - \ln(3) \quad = 2 + \ln\left(\frac{125}{3}\right)$

(6) (i) $I = \int x e^{2x}\, dx$

let $u = x$ $\quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1$ $\quad V = \frac{1}{2} e^{2x}$

$I = \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x}\, dx$

$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c$

(ii) $I = \int_{\frac{\pi}{4}}^{\pi} \sin^2 3x \, dx$

$= \int_{\frac{\pi}{4}}^{\pi} \frac{1}{2} - \frac{1}{2} \cos 2(3x) \, dx$

$= \int_{\frac{\pi}{4}}^{\pi} \frac{1}{2} - \frac{1}{2} \cos 6x \, dx$

$= \left[\frac{x}{2} - \frac{1}{12} \sin 6x \right]_{\frac{\pi}{4}}^{\pi}$

$= \frac{\pi}{2} - \frac{1}{12} \sin 6\pi - \frac{\pi}{8} + \frac{1}{12} \sin \frac{6\pi}{4}$

$= \frac{3\pi}{8} - 0 - \frac{1}{12}$

$= \frac{9\pi - 2}{24}$

(6) (iii) $\dfrac{x^2}{(x-4)^2(x-2)} \equiv \dfrac{A}{(x-4)^2} + \dfrac{B}{(x-4)} + \dfrac{C}{(x-2)}$

$$x^2 \equiv A(x-2) + B(x-4)(x-2) + C(x-4)^2$$

let $x = 2$

$$2^2 = 0 + 0 + C(2-4)^2$$
$$4 = 4C$$
$$C = 1$$

let $x = 4$

$$4^2 = 2A$$
$$16 = 2A$$
$$A = 8$$

let $x = 0$

$$0 = 8(-2) + B(-4)(-2) + 1(-4)^2$$

$$0 = -16 + 8B + 16$$

$$B = 0$$

$$\therefore I = \int_5^8 \frac{8}{(x-4)^2} + \frac{1}{x-2} \, dx$$

$$= \left[\frac{8}{-1}(x-4)^{-1} . 1 + \ln|x-2| \right]_5^8$$

$$= \left[\ln|x-2| - \frac{8}{x-4} \right]_5^8$$

$$= \ln(6) - \frac{8}{4} - \ln(3) + \frac{8}{1}$$

$$= \ln\left(\frac{6}{3}\right) + 6$$

$= 6 + \ln 2$ As required

(7)(i) $\dfrac{1-3x}{(1+2x)(1+x^2)} \equiv \dfrac{A}{1+2x} + \dfrac{Bx+C}{1+x^2}$

$1-3x \equiv A(1+x^2) + (Bx+C)(1+2x)$

$1-3x \equiv A + Ax^2 + Bx + 2Bx^2 + C + 2Cx$

Comparing coeficients

$x^2: \quad 0 = A + 2B \quad -(1)$

$x^1: \quad -3 = B + 2C \quad -(2)$

$x^0: \quad 1 = A + C \quad -(3)$

From (1) $\quad A = -2B \quad -(4)$

in (3) $\quad 1 = -2B + C$

$C = 1 + 2B \quad -(5)$

in (2) $\quad -3 = B + 2(1+2B)$

$-3 = B + 2 + 4B$

$-5 = 5B$

$B = -1$

in (4) $\quad A = +2$

in (5) $\quad C = -1$

$\therefore \quad f(x) = \dfrac{2}{1+2x} + \dfrac{-1x-1}{1+x^2}$

$= \dfrac{2}{1+2x} - \dfrac{(x+1)}{1+x^2}$

(7)(ii) $f(x) = 2(1+2x)^{-1} - (x+1)(1+x^2)^{-1}$

Binomial expansion of

$$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)(2x)^2}{2!} + \dots$$

$$\simeq 1 - 2x + 4x^2 \quad , \text{ valid for } |2x|<1 \ , \ |x|<\tfrac{1}{2}$$

$$(1+x^2)^{-1} \simeq 1 + (-1)(x^2) + \frac{(-1)(-2)(x^2)^2}{2!} + \dots$$

$$\simeq 1 - x^2 + \cancel{x^4} + \dots \quad , \text{ valid for } |x^2|<1 \ , \ |x|<1$$

only need up to order 2

$$\therefore f(x) \simeq 2(1-2x+4x^2) - (x+1)(1-x^2)$$

$$\simeq 2 - 4x + 8x^2 - (x - \cancel{x^3} + 1 - x^2)$$

$\simeq$ [illegible]

$$\simeq 2 - 4x + 8x^2 - x - 1 + x^2$$

$$\simeq \underline{1 - 5x + 9x^2} \quad \text{As required}$$

Valid for strictest of $|x| < \frac{1}{2}$ and $|x|<1$, ie $|x|<\frac{1}{2}$

(iii) @ small angles, $\sin\theta \simeq \theta$

$$\therefore I \simeq \int_0^{0.1} 1 - 5\theta + 9\theta^2 \, d\theta$$

$$\simeq \left[\theta - \frac{5\theta^2}{2} + 3\theta^3 \right]_0^{0.1}$$

$$= 0.1 - \frac{5(0.1)^2}{2} + 3(0.1)^3 - 0$$

$$= \underline{0.078}$$

⑧ (i) $f(x) = \dfrac{16+2x+15x^2}{(1+x^2)(2-x)} = \dfrac{A+Bx}{1+x^2} + \dfrac{C}{2-x}$

$16+2x+15x^2 \equiv (A+Bx)(2-x) + C(1+x^2)$

$x=2$

$16+2(2)+15(2)^2 = 0 + C(1+2^2)$

$16+4+60 = 5C$

$80 = 5C$

$C=16$

$x=0$

$16 = 2A + 16$

$A=0$

$x=1$

$16+2+15 = B(1) + 16(2)$

$33 = B+32$

$B=1$

(ii) $\therefore$ $I = \displaystyle\int_0^1 \frac{x}{1+x^2} + \frac{16}{2-x}\, dx$

$= \left[\frac{1}{2}\ln(1+x^2) + 16\ln(2-x).-1\right]_0^1$

$= \cancel{\ln (1+x^2)^{\frac{1}{2}}(2-x)^{16}}$

$= \cancel{\ln (2)^{\frac{1}{2}}(1) - \ln(1)(2)^{16}}$

$= \cancel{\ln \left(\frac{2}{2}\right)^{\frac{1}{2}}}$

$= \left[\frac{1}{2}\ln(1+x^2) - 16\ln(2-x)\right]_0^1$

$= \left(\frac{1}{2}\ln 2 - 16\ln(1)\right) - \left(\frac{1}{2}\ln(1) - 16\ln 2\right)$

$= \frac{1}{2}\ln 2 + 16\ln 2 = \frac{33}{2}\ln 2$

$$⑧(iii) \quad f(x) = x(1+x^2)^{-1} + 16(2-x)^{-1}$$

$$= x(1+x^2)^{-1} + 16\left[2^{-1}\left(1-\frac{x}{2}\right)^{-1}\right]$$

$$= x(1+x^2)^{-1} + 8\left(1-\frac{x}{2}\right)^{-1}$$

Binomial Expansion of

$$(1+x^2)^{-1} = 1 + (-1)(x^2) + \frac{(-1)(-2)(x^2)^2}{2!} + \dots$$

$$= 1 - x^2 + x^4 + \dots \quad \text{Valid for } |x^2|<1, \ |x|<1$$

$$\left(1-\frac{x}{2}\right)^{-1} = 1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(-\frac{x}{2}\right)^3$$

$$+ \frac{(-1)(-2)(-3)(-4)}{4!}\left(-\frac{x}{2}\right)^4 + \dots$$

$$\simeq 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \dots \quad \text{Valid for } \left|-\frac{x}{2}\right|<1, \ |x|<2$$

$$\therefore f(x) \simeq x\left[1 - x^2 + \cancel{x^4} + \dots\right] + 8\left[1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \dots\right]$$

$$\simeq x - x^3 + 8 + 4x + 2x^2 + x^3 + \frac{x^4}{2} + \dots$$

$$\simeq 8 + 5x + 2x^2 + \frac{x^4}{2} + \dots$$

Valid for strictest of $|x|<1$ and $|x|<2$, ie $|x|<1$

9 (i) $\dfrac{2u^2}{u^2-1}$

$$\begin{array}{r} 2 \\ u^2-1 \overline{)\, 2u^2 \quad} \\ 2u^2 - 2 \\ \hline 2 \end{array}$$

$\therefore \quad \dfrac{2u^2}{u^2-1} \equiv 2 + \dfrac{2}{u^2-1}$ As required

$\equiv 2 + \dfrac{2}{(u+1)(u-1)}$

$\equiv 2 + \dfrac{A}{u+1} + \dfrac{B}{u-1}$

$2 \equiv A(u-1) + B(u+1)$

$u=1 \quad 2 = 2B$

$B = 1$

$u=-1 \quad 2 = -2A$

$A = -1$

$\therefore \quad \dfrac{2u^2}{u^2-1} \equiv 2 - \dfrac{1}{u+1} + \dfrac{1}{u-1}$

(ii) $\dfrac{\sqrt{x}}{x-1}$ if $u = \sqrt{x}$, $x = u^2$

$\therefore \quad \dfrac{\sqrt{x}}{x-1} \equiv \dfrac{u}{u^2-1}$

Also if $I = \displaystyle\int_4^9 \dfrac{\sqrt{x}}{x-1}\,dx$ then $u = x^{1/2}$

$\dfrac{du}{dx} = \dfrac{1}{2}x^{-1/2}$

$dx = 2\sqrt{x}\,du$

$= 2u\,du$

change limits, when $x=4 \quad u=2$

$x=9 \quad u=3$

$\therefore \quad I = \displaystyle\int_2^3 \dfrac{u}{u^2-1} \cdot 2u\,du = \int_2^3 \dfrac{2u^2}{u^2-1}\,du$ As required

9ii) $I = \int_2^3 2 - \frac{1}{u+1} + \frac{1}{u-1} \, du$

$= \left[2u - \ln(u+1) + \ln(u-1) \right]_2^3$

$= \left[2u + \ln\left(\frac{u-1}{u+1}\right) \right]_2^3$

$= 6 + \ln\left(\frac{2}{4}\right) - 4 - \ln\left(\frac{1}{3}\right)$

$= 2 + \ln\left(\frac{3}{2}\right)$

$= \ln 3 - \ln 2 + 2$ As required

(iii) $\int_4^9 \frac{\ln(x-1)}{\sqrt{x}} \, dx = \int_4^9 \ln(x-1) \cdot x^{-1/2} \, dx$

let $u = \ln(x-1)$ $\quad \frac{dv}{dx} = x^{-1/2}$

$\frac{du}{dx} = \frac{1}{x-1}$ $\quad V = 2x^{+1/2}$

$I = \left[2\sqrt{x} \ln(x-1) \right]_4^9 - 2\int_4^9 \frac{\sqrt{x}}{x-1} \, dx$

$= 2(3)\ln 8 - 2(2)\ln 3 - 2\left[\ln 3 - \ln 2 + 2 \right]$

$= 6\ln 2^3 - 4\ln 3 - 2\ln 3 + 2\ln 2 - 4$

$= 18\ln 2 - 6\ln 3 - 4 + 2\ln 2$

$= 20\ln 2 - 6\ln 3 - 4$ As required.