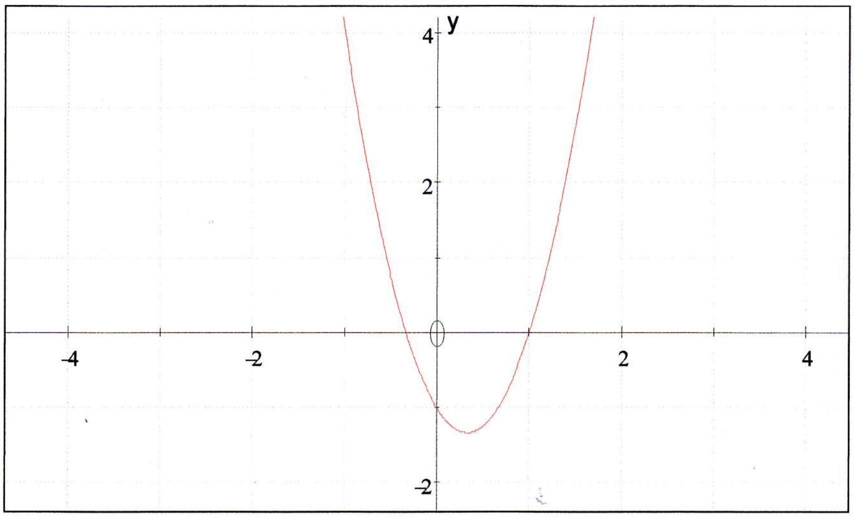
Classwork **06/10/08**

The Factor Theorem

Consider the function $f(x) = 3x^2 - 2x - 1$, the graph of $y = f(x)$ is shown below:



From the graph we can see that a solution to the equation $f(x) = 0$ lies at

$x = 1$ {where the curve crosses the x axis}

We could also have arrived at this result by factorising the quadratic expression:

$3x^2 - 2x - 1$ $-3x$ $-3, +1$

$3x^2 - 3x + 1x - 1 = 0$

$3x(x-1) + 1(x-1) = 0$

$(3x+1)(x-1) = 0$

either $3x + 1 = 0$ $x = -\frac{1}{3}$

or $x - 1 = 0$ $x = 1$

Also, we can see that $f(1) =$

$f(1) = 3(1)^2 - 2(1) - 1$
$= 3 - 2 - 1$
$= 0$

$f(-\frac{1}{3}) = 3(-\frac{1}{3})^2 - 2(-\frac{1}{3}) - 1$
$= \frac{3}{9} + \frac{2}{3} - 1 = \frac{3}{9} + \frac{6}{9} - \frac{9}{9} = \frac{9}{9} - \frac{9}{9} = 0$

This leads us to deduce that for any function $f(x)$, if $f(a) = 0$ then $(x - a)$ is a factor. This is known as the ***factor theorem*** and is an important instrument in being able to factorise cubic and higher polynomial expressions (cubic is as high as we need for Add Maths ☺).

Eg1   Factorise $x^3 - 2x^2 - x + 2$

Eg2   Show that $(x + 2)$ is a factor of $x^3 - 7x - 6$, hence solve the equation $x^3 - x + 6 = 0$

Eg3   Given that $(x - 4)$ is a factor of $x^3 - 2x^2 - 11x + 12$, find by long division the quadratic factor and hence factorise the expression completely.

Factor Theorem

eg/ let $f(x) = x^3 - 2x^2 - x + 2$

$$f(2) = (2)^3 - 2(2)^2 - (2) + 2$$
$$= 8 - 16 - 2 + 2$$
$$\neq 0$$

$$f(1) = (1)^3 - 2(1)^2 - (1) + 2$$
$$= 1 - 2 - 1 + 2$$
$$= 0$$

$\therefore$ $x = 1$ is a solution and $(x-1)$ is a factor

Now if $(x-1)$ is a factor, the other factor must be quadratic in order to produce an x^3 term.

$$\therefore \quad x^3 - 2x^2 - x + 2 \equiv (x-1)(Ax^2 + Bx + C)$$

Called an identity
LHS is identical to RHS

$$\equiv Ax^3 + Bx^2 + Cx - Ax^2 - Bx - C$$
$$\equiv Ax^3 + (B-A)x^2 + (C-B)x - C$$

Equating x^3 coefficients: $A = 1$

" x^2 " $B - A = -2$ $\quad B - 1 = -2$
$B = -2 + 1 = -1$

" x^1 " $C - B = -1$ $\quad C - (-1) = -1$
$C + 1 = -1$
$C = -2$

$$\therefore x^3 - 2x^2 - x + 2 \equiv (x-1)(x^2 - x - 2)$$
$$\equiv (x-1)(x-2)(x+1)$$

On a graph, looks like:

eg2 If $x+2$ is a factor, then $f(-2)=0$

$$f(-2) = (-2)^3 - 7(-2) - 6$$
$$= -8 + 14 - 6$$
$$= -14 + 14$$
$$= 0 \quad \text{as required.}$$

Now $(x+2)(Ax^2+Bx+C) \equiv x^3 + 0x^2 - 7x - 6$

$$Ax^3 + Bx^2 + Cx + 2Ax^2 + 2Bx + 2C \equiv x^3 + 0x^2 - 7x - 6$$

$$Ax^3 + (B+2A)x^2 + (2B+C)x + 2C \equiv x^3 + 0x^2 - 7x - 6$$

Comparing x^3 coefficients $A = 1$.

" x^2 " $B+2A = 0$ $B+2=0$ $B=-2$

" x^1 " $2B+C=-7$ $2(-2)+C=-7$

$-4+C=-7$

$C=-7+4=-3$

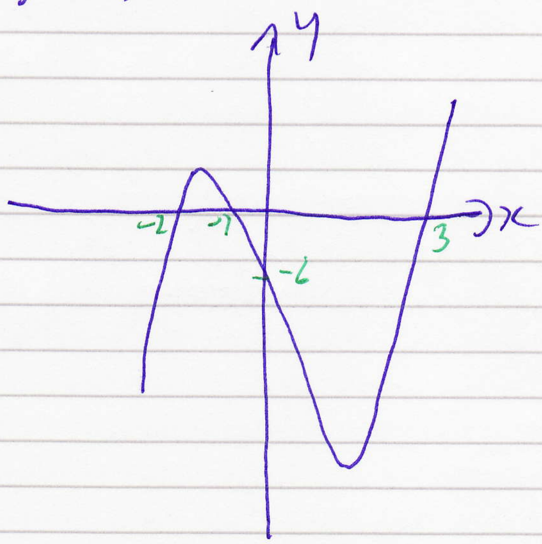
$\therefore f(x) = (x+2)(x^2-2x-3)$

$= (x+2)(x-3)(x+1)$

Now if $f(x)=0$ either $x+2=0$ or $x-3=0$ or $x+1=0$

$x=-2$ $x=3$ $x=-1$

On a graph, looks like:



eg3)

$$
\begin{array}{r|l}
 & \quad x^2 + 2x - 3 \\ \hline
x-4 & x^3 - 2x^2 - 11x + 12 \\
 & \underline{x^3 - 4x^2} \\
 & \qquad 2x^2 - 11x \\
 & \qquad \underline{2x^2 - 8x} \\
 & \qquad\qquad -3x + 12 \\
 & \qquad\qquad \underline{-3x + 12} \\
\end{array}
$$

$$\therefore x^3 - 2x^2 - 11x + 12 \equiv (x-4)(x^2+2x-3)$$

$$(x-4)(x+3)(x-1)$$

Exercise 1

Factorise the following expressions completely:

1. $x^3 - 6x^2 + 11x - 6$ $(x-1)(x-2)(x-3)$
2. $x^3 + 2x^2 - x - 2$ $(x-1)(x+1)(x+2)$
3. $x^3 + x^2 - 4x - 4$ $(x+2)(x+1)(x-2)$
4. $x^3 + 3x^2 - 4x - 12$ $(x+3)(x+2)(x-2)$
5. $x^3 - 7x - 6$ $(x+2)(x+1)(x-3)$
6. Show that $(2x - 1)$ is a factor of $2x^3 + x^2 + x - 1$ and find the quadratic factor using long division. x^2+x+1
7. Show that $3x^3 - 2x^2 + 3x - 2$ has $(3x - 2)$ as a factor and use long division to find the quadratic factor. x^2+1
8. Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ $x=3, x=\frac{1}{2}, x=-2$
9. Solve the equation $4x^3 - 20x^2 + 13x = 12$ $x=\frac{3}{2}, x=-\frac{1}{2}, x=4$