FP3 June 2006 Solutions

1

$$A^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$ is true for all positive integers n $\quad$ (S)

When $n=1$ $\quad A^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1+3) \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

so (S) is true for $n=1$

Assume (S) is true for $n=k$, say, that is

$$A^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

Need to show that (S) is true for $n=k+1$, that is

$$A^{k+1} = \begin{pmatrix} 1 & +\ k+1 & \frac{1}{2}[(k+1)^2+3(k+1)] \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+5k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now $\quad A^{k+1} = A^k A = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$2+k+\frac{1}{2}(k^2+3k)$
$= \frac{1}{2}(2k+4+k^2+3k)$
$= \frac{1}{2}(k^2+5k+4)$

$$= \begin{pmatrix} 1 & 1+k & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1+k & \frac{1}{2}(k^2+5k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

So if (S) is true for $n=k$, it is true for $n=k+1$

By Mathematical induction (S) is true for $n=1,2,\dots$ i.e all positive integers.

② (a) $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots\ldots$$

$f(x) = \cos 2x$	$f(\frac{\pi}{4}) = 0$
$f'(x) = -2\sin 2x$	$f'(\frac{\pi}{4}) = -2$
$f''(x) = -4\cos 2x$	$f''(\frac{\pi}{4}) = 0$
$f'''(x) = 8\sin 2x$	$f'''(\frac{\pi}{4}) = 8$
$f^{iv}(x) = 16\cos 2x$	$f^{iv}(\frac{\pi}{4}) = 0$
$f^{v}(x) = -32\sin 2x$	$f^{v}(\frac{\pi}{4}) = -32$

$$\Rightarrow \quad \cos 2x = -2(x-\tfrac{\pi}{4}) + \frac{8}{3!}(x-\tfrac{\pi}{4})^3 - \frac{32}{5!}(x-\tfrac{\pi}{4})^5 + \ldots$$

$$= -2(x-\tfrac{\pi}{4}) + \tfrac{4}{3}(x-\tfrac{\pi}{4})^3 - \tfrac{4}{15}(x-\tfrac{\pi}{4})^5 + \ldots$$

(b) Putting $x=1$ $$\cos 2 \approx -2(1-\tfrac{\pi}{4}) + \tfrac{4}{3}(1-\tfrac{\pi}{4})^3 - \tfrac{4}{15}(1-\tfrac{\pi}{4})^5 + \ldots$$

$$\approx -0.416147$$

③ (a) By De Moivre's Theorem $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ ①

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

so $$\operatorname{Im}\left[(\cos\theta + i\sin\theta)^5\right] = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5\cos^4\theta\sin\theta - 10\cos^2\theta(1-\cos^2\theta)\sin\theta + (1-\cos^2\theta)^2\sin\theta$$

$$= \sin\theta\left[5\cos^4\theta + 10\cos^4\theta - 10\cos^2\theta + \cos^4\theta + 1 - 2\cos^2\theta\right]$$

$$\Rightarrow \quad \sin 5\theta = \sin\theta\left[16\cos^4\theta - 12\cos^2\theta + 1\right]$$ as required.

(b) $$\sin 5\theta + \cos\theta\sin 2\theta = 0$$

$$\Rightarrow \sin\theta\left[16\cos^4\theta - 12\cos^2\theta + 1\right] + 2\sin\theta\cos^2\theta = 0$$

$$\Rightarrow \sin\theta\left[16\cos^4\theta - 10\cos^2\theta + 1\right] = 0$$

$$\Rightarrow \sin\theta = 0 \quad \text{or} \quad (8\cos^2\theta - 1)(2\cos^2\theta - 1) = 0$$

$$\Rightarrow \theta = 0$$

$\cos^2\theta = \frac{1}{8}$

$\cos\theta = \pm\frac{1}{2\sqrt{2}}$ or $\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = 1.209^c, 1.932$ $\qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

(4) (a) $\left(\frac{dx}{dt}\right)_0 \approx \frac{x_1 - x_0}{h} \Rightarrow 0.4 \approx \frac{x_1 - 0}{0.1}$

$\Rightarrow \underline{x_1 \approx 0.04}$

$\Rightarrow \underline{x_{0.1} \simeq 0.04}$

$$\left(\frac{d^2x}{dt^2}\right)_{0.1} = -3\sin x_{0.1} \approx \frac{x_{0.2} - 2x_{0.1} + 0}{(0.1)^2}$$

$\Rightarrow x_{0.2} \approx 2x_{0.1} + 0.01 \times -3\sin x_{0.1}$

$\underline{\underline{x_{0.2} \approx 0.0788}}$

$$\left(\frac{d^2x}{dt^2}\right)_{0.2} = -3\sin x_{0.2} \simeq \frac{x_{0.3} - 2x_{0.2} + x_{0.1}}{0.01}$$

$\Rightarrow x_{0.3} \simeq 2x_{0.2} - x_{0.1} - 0.03\sin x_{0.2}$

$\Rightarrow \underline{\underline{x_{0.3} \simeq 0.115}}$

(b) $x = f(t) \Rightarrow f(0) = 0$

$\frac{dx}{dt} = f'(t) \Rightarrow f'(0) = 0.4$

$\frac{d^2x}{dt^2} = -3\sin x \Rightarrow f''(0) = 0$

$\frac{d^3x}{dt^3} = -3\cos x \frac{dx}{dt} \Rightarrow f'''(0) = -3 \times 0.4$

$= \underline{\underline{-1.2}}$

$\Rightarrow f(t) = 0.4t - \frac{1.2}{3!}t^3 + \ldots$

$= \underline{\underline{0.4t - 0.2t^3 + \ldots}}$

(c) $x_{0.3} = f(0.3) \approx 0.4 \times 0.3 - 0.2 \times 0.3^3 + \ldots$

$\underline{\underline{\simeq 0.1146}}$

(5) (a)

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$
$$\Rightarrow 4 - 5\lambda + \lambda^2 + 2 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\Rightarrow (\lambda - 2)(\lambda - 3) = 0$$
$$\Rightarrow \underline{\underline{\lambda_1 = 2}} \text{ and } \underline{\underline{\lambda_2 = 3}}$$

(b)

$$|M| = 4\times 1 + 1\times 2$$
$$= \underline{\underline{6}}$$

$$\Rightarrow M^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} \end{pmatrix}}}$$

(c)

$$\begin{vmatrix} +\frac{1}{6} - \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} - \frac{1}{2} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} \end{vmatrix} = -\frac{1}{3}\times\frac{1}{6} + \frac{1}{3}\times\frac{1}{6} = 0$$ so $\frac{1}{2}$ is an eigenvalue of M^{-1}

$$\begin{vmatrix} \frac{1}{6} - \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} - \frac{1}{3} \end{vmatrix} = \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} = -\frac{1}{6}\times\frac{1}{3} + \frac{1}{6}\times\frac{1}{3} = 0$$ so $\frac{1}{3}$ is an eigenvalue of M^{-1}

(d)

$\underline{\underline{\lambda = 2}}$

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 4x - 2y &= 2x \\ \text{and} \quad x + y &= 2y \end{aligned}\right\} \text{both give } \underline{\underline{y = x}}$$

$\underline{\underline{\lambda = 3}}$

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 4x - 2y &= 3x \\ x + y &= 3y \end{aligned}\right\} \text{both give } \underline{\underline{y = \tfrac{1}{2}x}}$$

6 (a) Let $z = x + iy$

$$\Rightarrow |(x-6) + i(y+3)| = 3|(x+2) + i(y-1)|$$

$$\Rightarrow (x-6)^2 + (y+3)^2 = 9\left((x+2)^2 + (y-1)^2\right)$$

$$\Rightarrow x^2 - 12x + \cancel{36} + y^2 + 6y + \cancel{9} = 9x^2 + 36x + \cancel{36} + 9y^2 - 18y + \cancel{9}$$

$$\Rightarrow 8x^2 + 48x + 8y^2 - 24y = 0$$

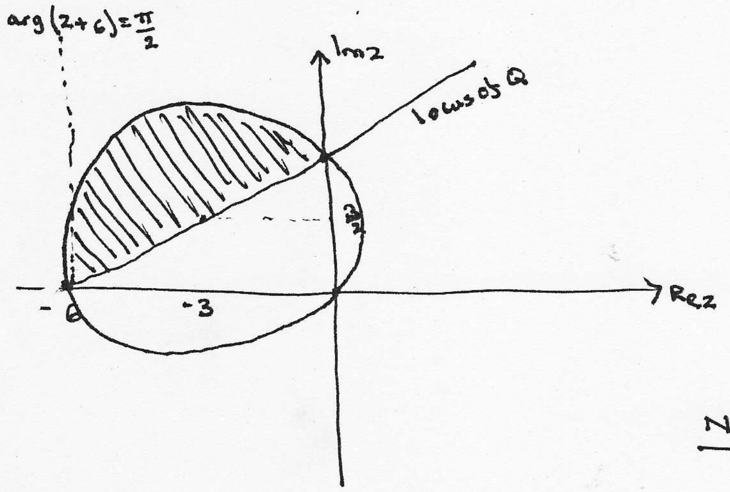
$$\rightarrow x^2 + 6x + y^2 - 3y = 0$$

$$\Rightarrow (x+3)^2 - 9 + \left(y - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} = 0$$

$$\Rightarrow (x+3)^2 + \left(y - \tfrac{3}{2}\right)^2 = \tfrac{45}{4}$$

so locus of z is a circle centre $\left(-3, \tfrac{3}{2}\right)$ and radius $\tfrac{3\sqrt{5}}{2}$.

(b)(c)



$x = 0$ $\quad 9 + \left(y - \tfrac{3}{2}\right)^2 = \tfrac{45}{4}$

$\left(y - \tfrac{3}{2}\right)^2 = \tfrac{9}{4}$

$y = 3$

$y = 0$ $\quad (x+3)^2 = 9$

$\underline{\underline{x = 0}}$ $\quad \underline{\underline{x = -6}}$

<u>NOTE</u> EDEXCEL SOLUTION IS WRONG!

7 (a) $\underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ $\qquad \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$

$$(\underline{b} - \underline{a}) \times (\underline{c} - \underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & -3 \\ 4 & -5 & -1 \end{vmatrix}$$

$$= \underline{\underline{-15\underline{i} - 10\underline{j} - 10\underline{k}}}$$

(b) The vectors $\begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ are normal to the plane.

Now if $\underline{n} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ then $\underline{a} \cdot \underline{n} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 7$

$\Rightarrow \underline{r} \cdot (3\underline{i} + 2\underline{j} + 2\underline{k}) = 7$ is the equation of Π_1.

7 (c) $\Pi_1: \quad 3x + 2y + 2z = 7$

$\Pi_2: \quad x + z = 3$

Put $x = \lambda$ $\quad z = 3 - \lambda$

and $\quad 3\lambda + 2y + 6 - 2\lambda = 7$

$\lambda + 2y + 6 = 7$

$2y = 1 - \lambda$

$y = \dfrac{1-\lambda}{2}$

so $\left(\lambda, \frac{1-\lambda}{2}, 3-\lambda\right)$ is any point on the line of intersection

$\dfrac{1-\lambda}{2} = y \Rightarrow \lambda = 1 - 2y$

$\lambda = \dfrac{y - \frac{1}{2}}{-\frac{1}{2}}$

$z = 3 - \lambda \Rightarrow \lambda = 3 - z$

$= \dfrac{z-3}{-1}$

$\Rightarrow$ The Cartesian equation of line is $\dfrac{x-0}{1} = \dfrac{y-\frac{1}{2}}{-\frac{1}{2}} = \dfrac{z-3}{-1} (= \lambda)$

so $(0, \frac{1}{2}, 3)$ is a point on the line and $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ is a direction vector.

$\Rightarrow$ vector equation of line is $\underline{r} = \frac{1}{2}\underline{j} + 3\underline{k} + \lambda(2\underline{i} - \underline{j} - 2\underline{k})$

or $\quad \left(\underline{r} - \left(\frac{1}{2}\underline{j} + 3\underline{k}\right)\right) \times (2\underline{i} - \underline{j} - 2\underline{k}) = \underline{0}$

(d) For shortest distance the angle between OP and Line must be a rightangle

$\Rightarrow \begin{pmatrix} \lambda \\ \frac{1-\lambda}{2} \\ 3-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$

$\Rightarrow 2\lambda + \dfrac{(\lambda - 1)}{2} - 6 + 2\lambda = 0$

$\Rightarrow 4\lambda + \lambda - 1 - 12 + 4\lambda = 0$

$\Rightarrow 9\lambda = 13$

$\Rightarrow \lambda = \frac{13}{9}$

so P is $\left(\frac{13}{9}, \frac{-2}{9}, \frac{14}{9}\right)$