FP3 JANUARY 2006 SOLUTIONS

① $\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right) = i$

so if $z^5 = i$

$\Rightarrow z^5 = \cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)$

$\Rightarrow z = \left[\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)\right]^{\frac{1}{5}}$

$\Rightarrow z = \cos\left(\frac{\pi}{10}+\frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{10}+\frac{2k\pi}{5}\right)$

$\Rightarrow z = \cos\left(\frac{(4k+1)\pi}{5}\right) + i\sin\left(\frac{(4k+1)\pi}{5}\right)$ for $k=0,1,2,3,4$.

② $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$ $\qquad$ $\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_1}{h^2}$

$\Rightarrow 2 \approx \frac{y_1 - y_{-1}}{0.2}$ $\qquad$ $2 + 3\times1^2\times2 \approx \frac{y_1 - 2 + y_{-1}}{(0.1)^2}$

$\Rightarrow \underline{y_1 - y_{-1} \approx 0.4}$ (1) $\qquad$ $8\times(0.1)^2 \approx y_1 + y_{-1} - 2$

$\Rightarrow \underline{y_1 + y_{-1} \approx 2.08}$ (2)

(2) − (1) gives $2y_{-1} \approx 1.68$

$\underline{y_{-1} \approx 0.84}$

so $y \approx 0.84$ at $x = 0.4$

③ (a) $A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$

$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ 2x+1 \end{pmatrix} = \begin{pmatrix} -4x+4x+2 \\ 2x-2x-1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

so the image of $y = 2x+1$ is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(b) $\underline{k=2} \Rightarrow A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow (2-\lambda)(1+\lambda) + 4 = 0$

$\Rightarrow 2 + \lambda - \lambda^2 + 4 = 0$

$\Rightarrow 6 + \lambda - \lambda^2 = 0$

$\lambda^2 - \lambda - 6 = 0$

$(\lambda-3)(\lambda+2) = 0$

$\underline{\lambda = 3}$ or $\underline{\lambda = -2}$

3(c) ....

$\underline{\lambda = 3}$ $$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x + 2y = 3x \Rightarrow 2y = x \Rightarrow \underline{\underline{y = \tfrac{1}{2}x}}$$
$$2x - y = 3y \Rightarrow \underline{\underline{y = \tfrac{1}{2}x}}$$

$\underline{\underline{\lambda = -2}}$ $$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad 2x + 2y = -2x \Rightarrow \underline{\underline{y = -2x}}$$
$$2x - y = -2y \Rightarrow \underline{\underline{y = -2x}}$$

so $y = \frac{1}{2}x$ and $y = -2x$ are invariant under T.

4 (a) A is singular $\Rightarrow$ det A = 0

$$\Rightarrow \quad k\begin{vmatrix} -1 & k \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & k \\ 9 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ 9 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad k \times -k + 9k - 18 = 0$$
$$\Rightarrow \quad k^2 - 9k + 18 = 0$$
$$\Rightarrow \quad (k-6)(k-3) = 0$$
$$\Rightarrow \quad \underline{\underline{k = 3}} \text{ and } \underline{\underline{k = 6}}$$

(b) $$A = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$$

$\underline{\text{minors}}$ $$\begin{pmatrix} -k & -9k & 9 \\ 2 & 18 & k-9 \\ k-2 & k^2 & -k \end{pmatrix}$$

$\underline{\text{cofactors}}$ $$\begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & 9-k \\ k-2 & -k^2 & -k \end{pmatrix}$$

$\underline{\text{transpose}}$ $$\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & 9-k & -k \end{pmatrix}$$

$$\underline{\underline{\text{so } A^{-1} = \frac{1}{-k^2+9k-18}\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & 9-k & -k \end{pmatrix}}}$$

5 (a) $$\sum_{r=1}^{n} r2^r = 2\{1+(n-1)2^n) \qquad (1)$$

assume (1) is true for $n=k$ and prove (1) is true for $n=k+1$

i.e $$\sum_{r=1}^{k+1} r2^r = 2\left[1+k2^{k+1}\right]$$

Now $$\sum_{r=1}^{k+1} r2^r = \sum_{r=1}^{k} r2^r + (k+1)2^{k+1}$$

$$= 2(1+(k-1)2^k) + (k+1)2^{k+1}$$

$$= 2 + (k-1)2^{k+1} + (k+1)2^{k+1}$$

$$= 2 + 2^{k+1}(k-1+k+1)$$

$$= 2 + 2k \times 2^{k+1}$$

$$= 2(1+k2^{k+1})$$ as required.

so if (1) is true for $n=k$ it is also true for $n=k+1$

when $n=1$ $$\sum_{r=1}^{n} r2^r = 1\times 2^1 = 2$$

and $$2(1+(n-1)2^n) = 2(1+0\times 2^1) = \underline{\underline{2}}$$

So (1) is true for $n=1$ and hence true by induction for all $n \in \mathbb{Z}^+$.

(b) $$\frac{d^n y}{dx^n} = (-1)^{n+1} \cdot \frac{(n-1)!\,3^n}{(2+3x)^n} \qquad (1)$$

when $n=1$ $\dfrac{dy}{dx} = \dfrac{3}{2+3x}$ and $\dfrac{d}{dx}(\ln(2+3x)) = \dfrac{3}{2+3x}$

so (1) is true for $n=1$.

Assume (1) is true for $n=k$ and prove (1) is true for $n=k+1$

i.e $$\frac{d^{k+1}y}{dx^{k+1}} = (-1)^{k+2}\frac{k!\,3^{k+1}}{(2+3x)^{k+1}}$$

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k x}{dx^k}\right) = \frac{d}{dx}\left[(-1)^{k+1}\frac{(k-1)!\,3^k}{(2+3x)^k}\right]$$

$$= (-1)^{k+1}(k-1)!\,3^k \frac{d}{dx}\left[(2+3x)^{-k}\right]$$

$$= (-1)^{k+1}(k-1)!\,3^k \times -k\times 3\times(2+3x)^{-k-1}$$

$$= \frac{(-1)^{k+2}k!\,3^{k+1}}{(2+3x)^{k+1}}$$ as required

so if (1) is true for $n=k$ it is also true for $n=k+1$

So (1) is true for $n=1$ by induction it is true for all $n \in \mathbb{Z}^+$.

(6) $$(1+2x)\frac{dy}{dx} = x + 4y^2$$

(a) Diff. w.r.t x
$$(1+2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 1 + 8y\frac{dy}{dx}$$
$$\Rightarrow \quad (1+2x)\frac{d^2y}{dx^2} = 1 + 8y\frac{dy}{dx} - 2y\frac{dy}{dx}$$
$$\Rightarrow \quad (1+2x)\frac{d^2y}{dx^2} = 1 + 2(4y-1)\frac{dy}{dx}$$

(b)
$$(1+2x)\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 2(4y-1)\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2$$
$$\Rightarrow (1+2x)\frac{d^3y}{dx^2} = 4(2y-1)\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2$$

(c)
$$y = \tfrac{1}{2} \quad x = 0$$
$$\left(\frac{dy}{dx}\right)_0 = 4 \times \tfrac{1}{4} = 1$$
$$\left(\frac{d^2y}{dx^2}\right)_0 = 1 + 8 \times \tfrac{1}{2} \times 1 - 2 \times 1$$
$$= 1 + 4 - 2$$
$$= \underline{\underline{3}}$$
$$\left(\frac{d^3y}{dx^3}\right)_0 = 4\left(2 \times \tfrac{1}{2} - 1\right) \times 3 + 8 \times 1^2$$
$$= \underline{\underline{8}}$$

$$\Rightarrow \quad \underline{\underline{y = \tfrac{1}{2} + x + \tfrac{3}{2}x^2 + \tfrac{4}{3}x^3 + \ldots}}$$

(7) (a) $\overrightarrow{RP} = \underline{p} - \underline{r} = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix}$ $\quad \overrightarrow{RQ} = \underline{q} - \underline{r} = \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 3 & -2-c \\ +1 & -1 & -1-c \end{vmatrix}$$
$$= [3(-1-c) + 1(2-c)]\underline{i} - [-4(-1-c) - 1(-2-c)]\underline{j} + \underline{k}$$
$$= [-3-3c-2-c]\underline{i} - [4+4c+2+c]\underline{j} + \underline{k}$$
$$= [-5-4c]\underline{i} - [5c+6]\underline{j} + \underline{k}$$

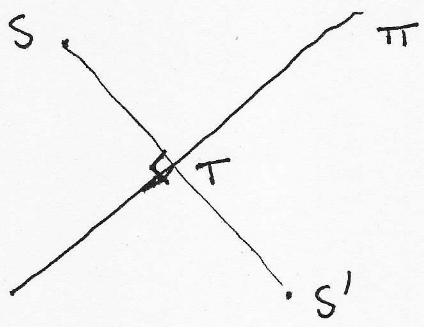
(b) $$-5-4c = 3 \Rightarrow -4c = 8 \Rightarrow \underline{\underline{c=-2}}$$
$$d = -5c - 6$$
$$d = 10 - 6$$
$$\underline{\underline{d = 4}}$$

(c) $\underline{n} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ is normal to Π.

$$\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = -3 + 12 - 2 = \underline{\underline{7}} \quad \Rightarrow \quad \underline{\underline{\underline{r} \cdot (3\underline{i} + 4\underline{j} + \underline{k}) = 7}}$$ (EQUATION OF Π)

(7) continued ...

(d)



Equation of line through S, T and S' is

$$\underline{r} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 1 + 3\lambda \\ 5 + 4\lambda \\ 10 + \lambda \end{pmatrix}$$

This line meets the plane where

$$\begin{pmatrix} 1 + 3\lambda \\ 5 + 4\lambda \\ 10 + \lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7$$

$$3 + 9\lambda + 20 + 16\lambda + 10 + \lambda = 7$$

$$26\lambda = -26$$

$$\underline{\underline{\lambda = -1}}$$

so T is the point $T(-2, 1, 9)$

$$\overrightarrow{ST} = \underline{t} - \underline{s} = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{TS'} = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$$

$$\Rightarrow \underline{s}' = \underline{t} + \overrightarrow{SS'}$$

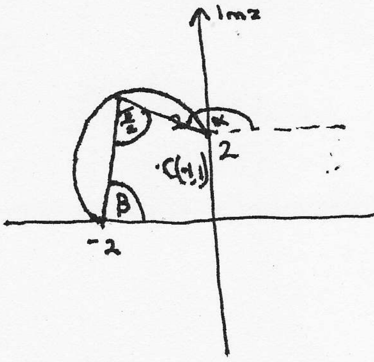
$$= \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -5 \\ -3 \\ 8 \end{pmatrix}}}$$

so S' is $\underline{\underline{(-5, -3, 8)}}$

8 (a) $\arg(z-2i) - \arg(z+2) = \frac{\pi}{2}$

let $\arg(z-2i) = \alpha$
and $\arg(z+2) = \beta$



Since $\alpha - \beta = \frac{\pi}{2}$

the locus of P is a semi-circle with $(-2,0)$ and $(0,2)$ at ends of diameter

$\Rightarrow$ centre is $(-1,1)$ and radius is

$r = \frac{1}{2}\sqrt{2^2+2^2}$

$r = \sqrt{2}$

Equation of circle is $(x+1)^2 + (y-1)^2 = 2$

(top half is locus of P).

OR ALGEBRAICALLY

If $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2}$

Then $\mathrm{Re}\left(\frac{z-2i}{z+2}\right) = 0$

If $z = x+iy$

$$\mathrm{Re}\left(\frac{z-2i}{z+2}\right) = \mathrm{Re}\left[\frac{x+(y-2)i}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy}\right]$$

$$= \frac{x(x+2)+y(y-2)}{(x+2)^2+y^2}$$

$$= 0$$

$\Rightarrow x(x+2) + y(y-2) = 0$

$\Rightarrow (x+1)^2 - 1 + (y-1)^2 - 1 = 0$

$(x+1)^2 + (y-1)^2 = 2$ circle centre $(-1,1)$ radius $\sqrt{2}$.

(b) $|z+1-i| = |z-(-1+i)|$

$=$ distance from centre of circle

$= \sqrt{2}$

(c) $w = \frac{2(1+i)}{z+2}$

$z+2 = \frac{2(1+i)}{w}$

$z = \frac{2(1+i)}{w} - 2$

$z = \frac{2(1+i) - 2w}{w}$

$$\frac{z-2i}{z+2} = \frac{2(1+i)-2w}{w} - 2i \Big/ \frac{2(1+i)-2w}{w} + 2$$

$$= \frac{2(1+i) - 2w - 2iw}{2(1+i) - 2w + 2w}$$

$$= \frac{2(1+i) - 2(1+i)w}{2(1+i)}$$

$$= 1 - w$$

$\Rightarrow \arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2} \Rightarrow \arg(1-w) = \frac{\pi}{2}$

$\Rightarrow \arg[-1(w-1)] = \frac{\pi}{2}$

$\Rightarrow \arg(-1) + \arg(w-1) = \frac{\pi}{2}$

$\Rightarrow \pi + \arg(w-1) = \frac{\pi}{2}$

$\Rightarrow \arg(w-1) = -\frac{\pi}{2}$