# FP2 JUNE 2007 SOLUTIONS

①

$$\int_1^3 \frac{1}{\sqrt{x^2+4x-5}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{(x+2)^2-9}}\,dx$$

$$= \int_3^5 \frac{1}{\sqrt{u^2-9}}\,du$$

$$= \left[\operatorname{arcosh}\frac{u}{3}\right]_3^5$$

$$= \operatorname{arcosh}\frac{5}{3} - \cancel{\operatorname{arcosh} 1}^{\,0}$$

$$= \operatorname{arcosh}\frac{5}{3}$$

$$= \ln\left[\frac{5}{3}+\sqrt{\left(\frac{5}{3}\right)^2-1}\right]$$

$$= \ln\left[\frac{5}{3}+\frac{4}{3}\right]$$

$$= \underline{\underline{\ln 3}}$$

Let $u = x+2$

$$\frac{du}{dx} = 1$$

$$\int du = \int dx$$

$x = 3 \Rightarrow u = 5$

$x = 1 \Rightarrow u = 3$

②

D: $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

$a = 5 \quad b = 3, \quad a > b$

$b^2 = a^2(1-e^2)$

$\frac{9}{25} = 1 - e^2$

$e^2 = \frac{16}{25}$

$\underline{\underline{e = \frac{4}{5}}}$

$\underline{\underline{\text{Foci at } (\pm 4, 0)}}$

E: $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$a = 2 \quad b = 3 \quad b > a$

$a^2 = b^2(1-e^2)$

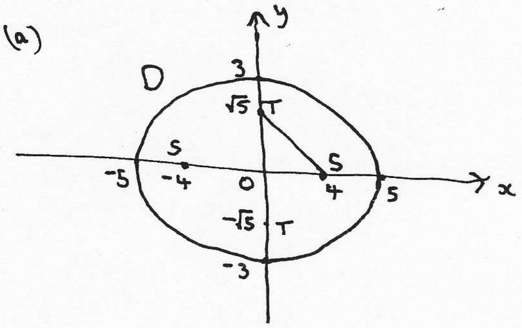
$\frac{4}{9} = 1 - e^2$

$e^2 = \frac{5}{9}$

$e = \frac{\sqrt{5}}{3}$

$\underline{\underline{\text{Foci at } (0, \pm\sqrt{5})}}$

(a)




$ST^2 = 4^2 + (\sqrt{5})^2$

$ST^2 = 21$

$\underline{\underline{ST = \sqrt{21}}}$

③

$$\frac{dy}{dx} = \frac{1}{4}\left(4x - \frac{1}{x}\right)$$
$$= \frac{4x^2-1}{4x}$$
$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{(4x^2-1)^2}{16x^2}$$
$$= \frac{16x^2+16x^4-8x^2+1}{16x^2}$$
$$= \frac{16x^4+8x^2+1}{16x^2}$$
$$= \left(\frac{4x^2+1}{4x}\right)^2$$
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{4x^2+1}{4x}$$
$$= x + \frac{1}{4x}$$
$$s = \int_{0.5}^{2} x + \frac{1}{4x}\ dx$$
$$= \left[\frac{x^2}{2} + \frac{1}{4}\ln x\right]_{0.5}^{2}$$
$$= \left[2 + \frac{1}{4}\ln 2\right] - \left[\frac{1}{8} + \frac{1}{4}\ln\frac{1}{2}\right]$$
$$= \frac{15}{8} + \frac{1}{4}\ln 2 - \ln 1 + \ln 2$$
$$= \underline{\underline{\frac{15}{8} + \frac{5}{4}\ln 2}}$$

④ (a) $\cosh(A-B) \equiv \dfrac{e^{A-B}+e^{B-A}}{2}$

$$\cosh A\cosh B - \sinh A\sinh B \equiv \left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right) - \left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$
$$\equiv \frac{\cancel{e^{A+B}} + e^{A-B} + e^{B-A} + \cancel{e^{-A-B}} - (\cancel{e^{A+B}} - e^{A-B} - e^{B-A} + \cancel{e^{-A-B}})}{4}$$
$$\equiv \frac{2e^{A-B}+2e^{B-A}}{4}$$
$$\equiv \frac{e^{A-B}+e^{B-A}}{2}$$
$$\equiv \underline{\underline{\cosh(A-B)}}$$

(b) $\cosh(x-1) \equiv \cosh x\cosh 1 - \sinh x\sinh 1$

$$\sinh x = \cosh x\left(\frac{e+e^{-1}}{2}\right) - \sinh x\left(\frac{e-e^{-1}}{2}\right)$$
$$2\sinh x = \cosh x(e+e^{-1}) - \sinh x(e-e^{-1})$$
$$2e\sinh x = \cosh x(e^2+1) - \sinh x(e^2-1)$$
$$(2e+e^2-1)\sinh x = \cosh x(e^2+1)$$
$$\underline{\underline{\tanh x = \frac{e^2+1}{2e+e^2-1}}}$$

⑤ (a)
$$x = t - \sin 2t$$
$$y = \cos 2t$$
$$\dot{x} = \frac{dx}{dt} = 1 - 2\cos 2t$$
$$\dot{y} = \frac{dy}{dt} = -2\sin 2t$$
$$\ddot{x} = \frac{d^2x}{dt^2} = 4\sin 2t$$
$$\ddot{y} = \frac{d^2y}{dt^2} = -4\cos 2t$$
$$\dot{x}^2+\dot{y}^2 = (1-2\cos 2t)^2 + 4\sin^2 2t$$
$$= 1 - 4\cos 2t + 4\cos^2 2t + 4\sin^2 2t$$
$$= \underline{\underline{5 - 4\cos 2t}}$$
$$\dot{x}\ddot{y} - \ddot{x}\dot{y} = -4\cos 2t(1-2\cos 2t) - 4\sin 2t \times -2\sin 2t$$
$$= -4\cos 2t + 8\cos^2 2t + 8\sin^2 2t$$
$$= \underline{8 - 4\cos 2t}$$
$$\Rightarrow \rho = \underline{\underline{\frac{(5-4\cos 2t)^{\frac{3}{2}}}{(8-4\cos 2t)}}}$$

(b) The least value of y is −1

$$\Rightarrow \cos 2t = -1$$
$$\Rightarrow 2t = \pi \qquad \text{for } 0 \le t < \pi$$
$$\underline{\underline{t = \frac{\pi}{2}}}$$
$$\Rightarrow \rho = \frac{(5-4\cos\pi)^{\frac{3}{2}}}{8-4\cos\pi}$$
$$\Rightarrow \rho = \frac{9^{\frac{3}{2}}}{12}$$
$$\Rightarrow \rho = \frac{27}{12}$$
$$\Rightarrow \rho = \underline{\underline{\frac{9}{4}}}$$

(7) (a) $y = \text{arsinh}(\sqrt{x})$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(\sqrt{x})^2}} \times \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1+x}}$$

$$= \frac{1}{2\sqrt{x(1+x)}}$$

when $x=4$ $\quad \frac{dy}{dx} = \frac{1}{2\sqrt{4\times5}}$

$$= \frac{1}{4\sqrt{5}}$$

(b) $\text{Area} = \int_0^4 \text{arsinh}(\sqrt{x})\, dx$

$x = \sinh^2\theta$

$\frac{dx}{d\theta} = 2\sinh\theta\cosh\theta$

$\int dx = \int 2\sinh\theta\cosh\theta\, d\theta$

$x=4 \quad \sinh\theta = 2$

$x=0 \quad \sinh\theta = 0$

$\sqrt{x} = \sinh\theta$

$\theta = \text{arsinh}\sqrt{x}$

$$= \int_{\text{arsinh}0}^{\text{arsinh}2} \theta\sinh 2\theta\, d\theta$$

$$= \int_0^{\text{arsinh}2} \theta\sinh 2\theta\, d\theta$$

$$= \left[\frac{1}{2}\theta\cosh 2\theta\right]_0^{\text{arsinh}2} - \frac{1}{2}\int_0^{\text{arsinh}2} \cosh 2\theta\, d\theta$$

$$= \left[\frac{1}{2}\theta\cosh 2\theta\right]_0^{\text{arsinh}2} - \frac{1}{4}\left[\sinh 2\theta\right]_0^{\text{arsinh}2}$$

$\cosh 2\theta = 1 + 2\sinh^2\theta$

$\frac{1}{2}\theta\cosh 2\theta = \frac{1}{2}\theta(1+2\sinh^2\theta)$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$= 2\sinh\theta\sqrt{(1+\sinh^2\theta)}$

$\text{arsinh}\,2 = \ln[2+\sqrt{2^2+1}]$

$= \ln[2+\sqrt{5}]$

$$= \left[\frac{1}{2}\text{arsinh}2\left(1+2(\sinh(\text{arsinh}2))^2\right)\right] - [0]$$

$$= \frac{9}{2}\text{arsinh}2 - \frac{1}{4}[2\times2\times\sqrt{5}]$$

$$= \frac{9}{2}\text{arsinh}2 - \sqrt{5}$$

$$= \frac{9}{2}\ln[2+\sqrt{5}] - \sqrt{5}$$

so $k = \frac{9}{2}$

(8) (a) $I_n = \int_0^8 x^n(8-x)^{\frac{1}{3}}\, dx$

$$= \left[-\frac{3}{4}x^n(8-x)^{\frac{4}{3}}\right]_0^8 + \frac{3}{4}n\int_0^8 x^{n-1}(8-x)^{\frac{4}{3}}\, dx$$

$$= \frac{3}{4}n\left[\int_0^8 x^{n-1}(8-x)(8-x)^{\frac{1}{3}}\, dx\right]$$

$$= \frac{3}{4}n\left[8\int_0^8 x^{n-1}(8-x)^{\frac{1}{3}}\, dx - \int_0^8 x^n(8-x)^{\frac{1}{3}}\, dx\right]$$

$$= 6n\int_0^8 x^{n-1}(8-x)^{\frac{1}{3}}\, dx - \frac{3}{4}n\int_0^8 x^n(8-x)^{\frac{1}{3}}\, dx$$

$$= 6nI_{n-1} - \frac{3}{4}nI_n$$

$\Rightarrow I_n = 6nI_{n-1} - \frac{3}{4}nI_n$

$\Rightarrow I_n\left(1+\frac{3}{4}n\right) = 6nI_{n-1}$

$\Rightarrow I_n(4+3n) = 24nI_{n-1}$

$\Rightarrow I_n = \left(\frac{24n}{4+3n}\right)I_{n-1}$ $\quad$ (1)

(b) $\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\, dx$

$$= \int_0^8 x^2(8-x)^{\frac{1}{3}}\, dx + 5\int_0^8 x(8-x)^{\frac{1}{3}}\, dx$$

$$= I_2 + 5I_1 \quad (2)$$

Now $I_2 = \frac{24}{5}I_1$ $\quad$ from (1)

$I_1 = \frac{24}{7}I_0$

Now $I_0 = \int_0^8 (8-x)^{\frac{1}{3}}\, dx$

$$= \left[-\frac{3}{4}(8-x)^{\frac{4}{3}}\right]_0^8$$

$$= [0] + \left[\frac{3}{4}\times 8^{\frac{4}{3}}\right]$$

$$= 12$$

so $\int_0^8 x(x+5)(8-x)^{\frac{1}{3}}\, dx$

$$= I_2 + 5I_1$$

$$= \frac{24}{5}I_1 + \frac{120}{7}I_0$$

$$= \frac{24}{5}\times\frac{24}{7}\times I_0 + \frac{120}{7}I_0$$

$$= \left(\frac{576}{35} + \frac{120}{7}\right)\times 12$$

$$= \frac{2016}{5}$$

(8) (a) Eq$^{n}$ of chord PQ

$$\text{Gradient of } PQ = \frac{2aq - 2ap}{aq^2 - ap^2}$$
$$= \frac{2a(q-p)}{a(q^2-p^2)}$$
$$= \frac{2\cancel{(q-p)}}{\cancel{(q-p)}(q+p)}$$
$$= \frac{2}{q+p}$$

So $y - 2ap = \frac{2}{q+p}(x - ap^2)$

$$(q+p)(y-2ap) = 2(x-ap^2)$$
$$qy - 2apq + py - 2ap^2 = 2x - 2ap^2$$
$$(q+p)y = 2x + 2apq$$
$$(q+p)y = 2(x + apq)$$

(b) $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$
$$y\frac{dy}{dx} = 2a$$
$$\frac{dy}{dx} = \frac{2a}{y}$$

At P $\frac{dy}{dx} = \frac{2a}{2ap} = \frac{1}{p}$

At Q $\frac{dy}{dx} = \frac{1}{q}$

Eq$^{n}$ of normal at P

$$y - 2ap = -p(x - ap^2) \quad (1)$$

Eq$^{n}$ of normal at Q

$$y - 2aq = -q(x - aq^2) \quad (2)$$

(1) − (2)

$$\Rightarrow 2aq - 2ap = q(x - aq^2) - p(x - ap^2)$$
$$\Rightarrow 2a(q-p) = (q-p)x - aq^3 + ap^3$$
$$\Rightarrow 2a(q-p) = (q-p)x - a(q^3 - p^3)$$
$$\Rightarrow 2a = x - a(p^2 + pq + q^2)$$

Since $q^3 - p^3 = (q-p)(q^2 + pq + p^2)$

$$\Rightarrow x = 2a + a(p^2 + pq + q^2)$$
$$x = a(p^2 + pq + q^2 + 2)$$

Sub into (1)

$$y = 2ap - p(a(p^2 + pq + q^2 + 2) - ap^2)$$
$$y = 2ap - p[apq + aq^2 - 2a]$$
$$y = \cancel{2ap} - ap^2q - apq^2 - \cancel{2ap}$$
$$y = -apq(p+q)$$

R is $(a(p^2 + pq + q^2 + 2), -apq(p+q))$

If PQ passes through $(5a, 0)$ then

$$(p+q)\times 0 = 2(5a + apq)$$
$$0 = 10a + 2apq$$
$$-10a = 2apq$$
$$-5 = pq$$

so R becomes

$$R(a(p^2 + q^2 - 3), 5a(p+q))$$

If R is $(x, y)$

$$y^2 = 25a^2(p+q)^2$$
$$y^2 = 25a^2(p^2 + 2pq + q^2)$$
$$y^2 = 25a^2(p^2 + q^2 - 10) \quad (3)$$

Now $x = a(p^2 + q^2 - 3)$

$$\Rightarrow \frac{x}{a} = p^2 + q^2 - 3$$
$$\Rightarrow \frac{x}{a} - 7 = p^2 + q^2 - 10$$

Sub into (3) $y^2 = 25a^2\left(\frac{x}{a} - 7\right)$

$$y^2 = 25ax - 175a^2$$