Ex 2B

(1) $$\sum_{r=1}^{13} r^2 = \frac{1}{6}(13)(14)(27) = 819$$

(2) $$\sum_{r=4}^{11} r^3 = \sum_{r=1}^{11} r^3 - \sum_{r=1}^{3} r^3$$

$$= \frac{1}{4}(11)^2(12)^2 - \frac{1}{4}(3)^2(4)^2$$

$$= 4320$$

(3) $$\sum_{r=11}^{24} r(r+1) = \sum_{r=11}^{24} r^2 + r = \sum_{r=12}^{24} r^2 + \sum_{r=11}^{24} r$$

$$= \sum_{r=1}^{24} r^2 - \sum_{r=1}^{10} r^2 + \sum_{r=1}^{24} r - \sum_{r=1}^{10} r$$

$$= \frac{1}{6}(24)(25)(49) - \frac{1}{6}(10)(11)(21) + \frac{1}{2}(24)(25) - \frac{1}{2}(10)(11)$$

$$= 4900 - 385 + 300 - 55$$

$$= \cancel{4628}\ 4760$$

(4) $$\sum_{r=1}^{19} r(r+4) = \sum_{r=1}^{19} r^2 + 4r = \sum_{r=1}^{19} r^2 + 4\sum_{r=1}^{19} r$$

$$= \frac{1}{6}19(20)(39) + 4 \times \frac{1}{2}19(20) = 2470 + 760 = 3230$$

(5) $$\cancel{\sum_{r=1}^{20} \frac{1}{r(r+1)} = \frac{1}{\sum_{r=1}^{20} r^2 + r}}$$

Now $$\cancel{\sum_{r=1}^{20} r^2 + \sum_{r=1}^{20} r = \frac{1}{6}20(21)(41) + \frac{1}{2}20(21) = 2870 + 210 = 3080}$$

⑤ $\displaystyle\sum_{r=1}^{20} \frac{1}{r(r+1)}$

using partial fractions $\displaystyle\frac{1}{r(r+1)} \equiv \frac{A}{r} + \frac{B}{r+1}$

using cover up $A = 1 \quad B = -1$

$$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)} \equiv \sum_{r=1}^{n} \frac{1}{r} - \frac{1}{(r+1)} \Rightarrow f(r) - f(r+1)$$

$r=1 \qquad \displaystyle\frac{1}{1(2)} = \frac{1}{1} - \cancel{\frac{1}{2}}$

$r=2 \qquad \displaystyle\frac{1}{2(3)} = \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$r=3 \qquad \displaystyle\frac{1}{3(4)} = \cancel{\frac{1}{3}} - \frac{1}{4}$

$r=n-1 \qquad \displaystyle\frac{1}{(n-1)n} = \frac{1}{n-1} - \cancel{\frac{1}{n}}$

$r=n \qquad \displaystyle\frac{1}{n(n+1)} = \cancel{\frac{1}{n}} - \frac{1}{n+1}$

adding. $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

$$\therefore \sum_{r=1}^{20} \frac{1}{r(r+1)} = \frac{20}{21}$$

⑥ $\displaystyle\sum_{r=3}^{16} (r+2)^3 = \sum_{r=3}^{16} (r+2)(r^2+4r+4) = \sum_{r=3} r^3 + 6r^2 + 12r + 8$

$$= \sum_{r=1}^{16} r^3 - \sum_{r=1}^{2} r^3 + 6\sum_{r=1}^{16} r^2 - 6\sum_{r=1}^{2} r^2 + 12\sum_{r=1}^{16} r - 12\sum_{r=1}^{2} r + 16(8) - 2(8)$$

$$= \frac{1}{4} 16^2(17)^2 - \frac{1}{4} 2^2(3)^2 + 6\times\frac{1}{6}16(17)(33) - 6\times\frac{1}{6} 2(3)(5) + 12\times\frac{1}{2} 16(17) - 12\times\frac{1}{2} 2(3) + 128 - 16$$

$$= 18496 - 9 + 8976 - 30 + 1632 - 36 + 128 - 16 = 29141$$

⑦ $\sum_{r=1}^{14}\left(\frac{3}{4}\right)^r = \frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots$

S_n terms of GP

$S_n = \frac{r(r^n-1)}{r-1}$

GP 1^{st} term $a: \frac{3}{4}$ $r=\frac{3}{4}$ $n=14$

$S = \frac{\frac{3}{4}\left(\left(\frac{3}{4}\right)^{14}-1\right)}{\frac{3}{4}-1} = 2.95$ (3 s.f.)

⑧ $\sum_{r=1}^{20}\frac{1}{(r+3)(r+6)}$

using partial fractions $\frac{1}{(r+3)(r+6)} = \frac{A}{r+3}+\frac{B}{r+6} = \frac{1}{3(r+3)} - \frac{1}{3(r+6)}$

$\neq f(r)-f(r+1)$??

$r=1$ $\frac{1}{(4)(7)} = \frac{1}{12} - \frac{1}{21}$

$r=2$ $\frac{1}{(5)(8)} = \frac{1}{15} - \frac{1}{24}$

$r=3$ $\frac{1}{(6)(9)} = \frac{1}{18} - \frac{1}{27}$

$r=4$ $\frac{1}{(7)(10)} = \frac{1}{21} - \frac{1}{30}$

overleaf

⑧ $$\sum_{r=1}^{20} \frac{1}{(r+3)(r+6)}$$

Using PF's: $$\frac{1}{(r+3)(r+6)} = \frac{A}{r+3} + \frac{B}{r+6} = \frac{1}{3(r+3)} - \frac{1}{3(r+6)}$$

$r=1$ $$\frac{1}{(4)(7)} = \frac{1}{3(4)} - \frac{1}{3(7)}$$

$r=2$ $$\frac{1}{(5)(8)} = \frac{1}{3(5)} - \frac{1}{3(8)}$$

$r=3$ $$\frac{1}{(6)(9)} = \frac{1}{3(6)} - \frac{1}{3(9)}$$

$r=4$ $$\frac{1}{(7)(10)} = \frac{1}{3(7)} - \frac{1}{3(10)}$$

$r=5$ $$\frac{1}{(8)(11)} = \frac{1}{3(8)} - \frac{1}{3(11)}$$

$r=n-2$ $$\frac{1}{(n+1)(n+4)} = \frac{1}{3(n+1)} - \frac{1}{3(n+4)}$$

$r=n-1$ $$\frac{1}{(n+2)(n+5)} = \frac{1}{3(n+2)} - \frac{1}{3(n+5)}$$

$r=n$ $$\frac{1}{(n+3)(n+6)} = \frac{1}{3(n+3)} - \frac{1}{3(n+6)}$$

Adding. $$\sum_{r=1}^{20} \frac{1}{(r+3)(r+6)} = \frac{1}{3(4)} + \frac{1}{3(5)} + \frac{1}{3(6)} - \frac{1}{3(n+4)} - \frac{1}{3(n+5)} - \frac{1}{3(n+6)}$$

$$= \frac{1}{12} + \frac{1}{15} + \frac{1}{18} - \frac{1}{3(24)} - \frac{1}{3(25)} - \frac{1}{3(26)}$$

$$= \frac{1}{12} + \frac{1}{15} + \frac{1}{18} - \frac{1}{72} - \frac{1}{75} - \frac{1}{78}$$

$$= \frac{1291}{7800} \quad (0.1655)$$

(9) $\sum_{r=4}^{16} (2r-1)^3 = \sum_{r=4}^{16} (2r-1)(4r^2-4r+1) = \sum_{r=4}^{16} 8r^3 - 12r^2 + 6r - 1$

$= 8\sum_{r=1}^{16} r^3 - 8\sum_{r=1}^{3} r^3 - 12\left[\sum_{r=1}^{16} r^2 - \sum_{r=1}^{3} r^2\right] + 6\left[\sum_{r=1}^{16} r - \sum_{r=1}^{3} r\right] - \left[16(1) - 3(1)\right]$

$= 8\times\frac{1}{4}16^2(17)^2 - 8\times\frac{1}{4}3^2(4)^2 - 12\left[\frac{1}{6}16(17)(33) - \frac{1}{6}3(4)(7)\right] + 6\left[\frac{1}{2}16(17) - \frac{1}{2}3(4)\right] - 13$

$= 147968 - 288 - 12\underbrace{(1496 - 14)}_{17784} + 6\underbrace{(136 - 6)}_{780} - 13$

$= 130663$

(10) $\sum_{r=3}^{23} r(r+1)(r+2) = \sum_{r=3}^{23} r(r^2+3r+2) = \sum_{r=3}^{23} r^3 + 3r^2 + 2r$

$= \left(\sum_{r=1}^{23} r^3 - \sum_{r=1}^{2} r^3\right) + 3\left(\sum_{r=1}^{23} r^2 - \sum_{r=1}^{2} r^2\right) + 2\left(\sum_{r=1}^{23} r - \sum_{r=1}^{2} r\right)$

$= \left(\frac{1}{4}23^2(24)^2 - \frac{1}{4}2^2 3^2\right) + 3\left(\frac{1}{6}(23)(24)(47) - \frac{1}{6}2(3)(5)\right) + 2\left(\frac{1}{2}23(24) - \frac{1}{2}(2)(3)\right)$

$= 76176 - 9 + 3\underbrace{(4324 - 5)}_{12957} + 2\underbrace{(276 - 3)}_{546}$

$= 89670$

(11) $\displaystyle\sum_{r=1}^{n}(2r-1)^2 \equiv \frac{1}{3}n(4n^2-1)$

$$\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}4r^2-4r+1 = 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + n(1)$$

$$= 4\left[\frac{1}{6}n(n+1)(2n+1)\right] - 4\left[\frac{1}{2}n(n+1)\right] + n$$

$$= \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$$

$$= \frac{n}{3}\left[\frac{2}{\cancel{3}}(2n^2+3n+1) - 6n^{\cancel{2}} - 6 + 3\cancel{n}\right]$$

$$= \frac{n}{3}\left[4n^2+6n+2-6n-6+3\right]$$

$$= \frac{n}{3}\left[4n^2-1\right] = \frac{1}{3}n(4n^2-1) \text{ as required.}$$

(12) $\displaystyle\sum_{r=1}^{n}r(2+r) \equiv \frac{1}{6}n(n+1)(2n+7)$

$$\sum_{r=1}^{n}r(2+r) = \sum_{r=1}^{n}r^2+2r = \sum_{r=1}^{n}r^2 + 2\sum_{r=1}^{n}r$$

$$= \frac{1}{6}n(n+1)(2n+1) + 2\frac{1}{2}n(n+1)$$

$$\cancel{=\frac{1}{6}n\Big[} = \frac{1}{6}n(n+1)\left[2n+1+6\right] = \frac{1}{6}n(n+1)(2n+7) \text{ as reqd.}$$

(13) $$\sum_{r=1}^{20} \frac{1}{4r^2-1} = \sum_{r=1}^{20} \frac{1}{(2r+1)(2r-1)}$$

$$\frac{1}{(2r+1)(2r-1)} = \frac{A}{2r+1} + \frac{B}{2r-1} \qquad A = -\frac{1}{2} \quad B = \frac{1}{2}$$

$$\therefore \sum_{r=1}^{20} \frac{1}{2(2r-1)} - \frac{1}{2(2r+1)}$$

$r=1$ $$\frac{1}{(2(1)+1)(2(1)-1)} = \frac{1}{2(1)} - \cancel{\frac{1}{2(3)}}$$

$r=2$ $$\frac{1}{(2(2)+1)(2(2)-1)} = \cancel{\frac{1}{2(3)}} - \cancel{\frac{1}{2(5)}}$$

$r=3$ $$\frac{1}{(2(3)+1)(2(3)-1)} = \cancel{\frac{1}{2(5)}} - \cancel{\frac{1}{2(7)}}$$

$r=4$ $$\frac{1}{(2(4)+1)(2(4)-1)} = \cancel{\frac{1}{2(7)}} - \frac{1}{2(9)}$$

$r=n-1$ $$\frac{1}{(2(n-1)+1)(2(n-1)-1)} = \frac{1}{2(2(n-1)-1)} - \frac{1}{2(2(n-1)+1)} = \frac{1}{2(2n-3)} - \cancel{\frac{1}{2(2n-1)}}$$

$r=n$ $$\frac{1}{(2n+1)(2n-1)} = \cancel{\frac{1}{2(2n-1)}} - \frac{1}{2(2n+1)}$$

adding $$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r-1)} = \frac{1}{2} - \frac{1}{2(2n+1)} = \frac{2n+1-1}{2(2n+1)} = \frac{\cancel{2}n}{\cancel{2}(2n+1)}$$

$$= \frac{n}{2n+1}$$

Now when $n=20$ $= \frac{20}{41}$

(14) $$\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$$

$$= \frac{1}{6}(2n)(2n+1)(2(2n)+1) - \frac{1}{6}(n-1)((n-1)+1)(2(n-1)+1)$$

$$= \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}(n-1)(n)(2n-1)$$

$$= \frac{n}{6}\left[2(2n+1)(4n+1) - (n-1)(2n-1)\right]$$

$$= \frac{n}{6}\left[16n^2+12n+2-2n^2+3n-1\right]$$

$$= \frac{n}{6}\left[14n^2+15n+1\right]$$

$$= \frac{n}{6}(14n+1)(n+1)$$

(15) $$f(r) = \frac{1}{r(r+1)}$$

$$f(r) - f(r+1) = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+1+1)} = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= \cancel{\tfrac{1}{r}}\frac{(r+2)-r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$$ As required.

Now $$\sum_{r=5}^{25} \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\sum_{r=5}^{25} \frac{2}{r(r+1)(r+2)}$$

(15) cont'd now if $\dfrac{2}{r(r+1)(r+2)} = \dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}$

$r=1$ $\quad \dfrac{2}{1(2)(3)} = \dfrac{1}{1(2)} - \dfrac{1}{\cancel{(2)(3)}}$

$r=2$ $\quad \dfrac{2}{2(3)(4)} = \dfrac{1}{\cancel{2(3)}} - \dfrac{1}{\cancel{(3)(4)}}$

$r=3$ $\quad \dfrac{2}{3(4)(5)} = \dfrac{1}{\cancel{3(4)}} - \dfrac{1}{\cancel{(4)(5)}}$

$r=4$ $\quad \dfrac{2}{4(5)(6)} = \dfrac{1}{\cancel{4(5)}} - \dfrac{1}{5(6)}$

$r=n-1$ $\quad \dfrac{2}{(n-1)(n)(n+1)} = \dfrac{1}{(n-1)(n)} - \dfrac{1}{\cancel{n(n+1)}}$

$r=n$ $\quad \dfrac{2}{n(n+1)(n+2)} = \dfrac{1}{\cancel{n(n+1)}} - \dfrac{1}{(n+1)(n+2)}$

adding. $\displaystyle\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)} = \frac{(n+1)(n+2)-2}{2(n+1)(n+2)} = \frac{n(n+3)}{2(n+1)(n+2)}$

Now $\displaystyle\frac{1}{2}\sum_{r=5}^{25} \frac{2}{r(r+1)(r+2)} = \frac{1}{2}\left[\sum_{r=1}^{25} \frac{2}{r(r+1)(r+2)} - \sum_{r=1}^{4} \frac{2}{r(r+1)(r+2)}\right]$

$$= \frac{1}{2}\left[\frac{25(28)}{2(26)(27)} - \frac{4(7)}{2(5)(6)}\right]$$

$$= \frac{1}{2}\left[\frac{700}{1404} - \frac{28}{60}\right]$$

$$= \frac{28}{1755}$$

(16) $$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$$

P.F. $$\frac{1}{(r+1)(r+2)} = \frac{A}{r+1} + \frac{B}{r+2} \qquad A=1 \quad B=-1$$

$$\therefore \quad \frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$$

$r=1$ $$\frac{1}{(2)(3)} = \frac{1}{2} - \cancel{\frac{1}{3}}$$

$r=2$ $$\frac{1}{(3)(4)} = \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$$

$r=3$ $$\frac{1}{(4)(5)} = \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$$

$r=4$ $$\frac{1}{(5)(6)} = \cancel{\frac{1}{5}} - \frac{1}{6}$$

$r=n-1$ $$\frac{1}{n(n+1)} = \frac{1}{n} - \cancel{\frac{1}{n+1}}$$

$r=n$ $$\frac{1}{(n+1)(n+2)} = \cancel{\frac{1}{n+1}} - \frac{1}{n+2}$$

all $\therefore$ $$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+2} = \frac{n+2-2}{2(n+2)} = \frac{n}{2(n+2)} \quad \text{as req}^{d}$$

(17) $$\left(\overset{1}{2} + \overset{2}{4} + \overset{3}{6} + \overset{4}{8} + \overset{5}{10} + \ldots + \overset{100}{200}\right) - \left(\overset{1}{3} + \overset{2}{6} + \overset{3}{9} + \ldots + \overset{66}{198}\right)$$

$$-\left(\overset{1}{6} + \overset{2}{12} + \overset{3}{18} + \overset{4}{24} + \ldots \overset{33}{198}\right)$$

× all numbers that are multiples of 3 rather than even multiples of 3.

$$\sum_{r=1}^{100} 2r - \sum_{r=1}^{\cancel{66}\ 33} \cancel{3}6r$$

$$2 \times \frac{1}{2}100(101) - \cancel{3}6 \times \frac{1}{2}\cancel{66(67)}\ 33(34)$$

$$10100 - \cancel{6633}\ 3366$$

$$= \underline{6734}$$

(18)

$$\sum_{r=1}^{100} 2r^2 - \sum_{r=1}^{200} r^2$$

$$2\sum_{r=1}^{100} r^2 - \sum_{r=1}^{200} r^2$$

$$2 \times \frac{1}{6}(100)(101)(201) - \frac{1}{6}(200)(201)(401)$$

$$= 676700 - 2686700$$

$$= -2010000$$

(19)

$$1^2 - 2^2 + 3^2 - 4^2 + \dots - (2n)^2$$

$$= (1^2 + 3^2 + 5^2 + \dots + (2n-1)^2) - (2^2 + 4^2 + 6^2 + \dots + (2n)^2)$$

$$= \sum_{r=1}^{n} (2r-1)^2 - (2r)^2$$

$$= \sum_{r=1}^{n} 4r^2 - 4r + 1 - 4r^2 = \sum_{r=1}^{n} 1 - 4\sum_{r=1}^{n} r$$

$$= n - 4 \times \frac{1}{2} n(n+1)$$

$$= n - 2n^2 - 2n$$

$$= -2n^2 - n$$

$$= -n(2n+1)$$

(20) $U_r = r(2r+1) + 2^{r+2}$

$$\sum_{r=1}^{n} U_r = \sum_{r=1}^{n} 2r^2 + r + 2^{r+2}$$

$$= 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r + \sum_{r=1}^{n} 2^{r+2} \Rightarrow 2^r.2^2 = 4.2^r$$

$$= 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 2^r$$

g.p. $a=2$ $r=2$ $S = \frac{a(r^n-1)}{r-1}$

$S = \frac{2(2^n-1)}{2-1} = 2^1.2^n - 2$

$= 2^{n+1} - 2$

$$= 2.\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) + 4(2^{n+1}-2)$$

$$= \frac{n(n+1)}{6}\left[2(2n+1) + 3\right] + 2^2 2^{n+1} - 8$$

$$= \frac{n}{6}(n+1)(4n+5) + 2^{n+3} - 8.$$