Ex 3D

(1) (a) let the square root be $a+ib$

then $(a+ib)^2 = 5+12i$

$$a^2 + 2abi + b^2i^2 = 5+12i$$

$$(a^2-b^2) + 2abi = 5+12i$$

Comparing real & imaginary parts

$$a^2-b^2 = 5 \quad -(1)$$

$$2ab = 12 \quad -(2)$$

from (2) $a = \frac{6}{b} \quad -(3)$

in (1) $\left(\frac{6}{b}\right)^2 - b^2 = 5$

$$\frac{36}{b^2} - b^2 = 5$$

$$36 - b^4 = 5b^2$$

$$b^4 + 5b^2 - 36 = 0$$

$$(b^2-4)(b^2+9) = 0$$

$\therefore$ either $b^2 = 4$ or $b^2 = -9$ $\times$ $b \in \mathbb{R}$

$b = \pm 2$

in (3) $a = \pm 3$

Hence square root of $5+12i$ are $3+2i$ & $-3-2i$

①(e) let $(a+bi)^2 = 1-(4\sqrt{3})i$

$a^2 - b^2 + 2abi = 1 - (4\sqrt{3})i$

equate real & imaginary.

$a^2 - b^2 = 1$ — ①

$2ab = -4\sqrt{3}$ — ②

from ② $a = \frac{-2\sqrt{3}}{b}$ — ③

in ① $\left(\frac{-2\sqrt{3}}{b}\right)^2 - b^2 = 1$

$\frac{12}{b^2} - b^2 = 1$

$12 - b^4 = b^2$

$b^4 + b^2 - 12 = 0$

$(b^2+4)(b^2-3) = 0$

$\therefore$ either $b^2 = -4$ ✗ $b \in \mathbb{R}$

or $b^2 = 3$ $b = \pm\sqrt{3}$

in ③ $b = +\sqrt{3}$ $a = -2$
$b = -\sqrt{3}$ $a = 2$

$\therefore$ roots are $-2+\sqrt{3}i$ and $2-\sqrt{3}i$

(2) d $x+4y+xyi = 12-16i$

equate real + imag

$x+4y = 12$ — (1)

$xy = -16$ — (2)

from (1) $x = 12-4y$ — (3)

~~in (2) $x(12-4y) = -16$~~

(3) in (2) $y(12-4y) = -16$

$12y - 4y^2 = -16$

$4y^2 - 12y - 16 = 0$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$\therefore$ either $y=4$ or $y=-1$

in (3) $x = 12-4(+4)$ or $x = 12-4(-1)$

$= 12-16$ $\quad = 12+4$

$= -4$ $\quad = 16$

$\therefore$ either $(16,-1)$ or $(-4,4)$.

(e) $2x - y + (y-4)i = 0$

$2x - y = 0$ — (1)

$y - 4 = 0$ — (2)

from (2) $y = 4$

in (1) $2x - 4 = 0$

$x = 2$

$(2,4)$

(3) (a) $(1+5i)A - 2B = 3+7i$

$A + 5Ai - 2B = 3+7i$

$A - 2B + 5Ai = 3+7i$

Equate real + imag

$A - 2B = 3$ — (1)

$5A = 7$ — (2)

from (1) $A = 3+2B$ — (3)

in (2) $5(3+2B) = 7$

in $15 + 10B = 7$

$10B = -8$

$B = \frac{-8}{10} = \frac{-4}{5}$

in (3) $A = 3 + 2 \times \frac{-4}{5} = 3 - \frac{8}{5} = \frac{7}{5}$

(b) let $A = x+iy$ $B = x-iy$

$(1+5i)(x+iy) - 2(x-iy) = 3+7i$

$x + \cancel{i}yi + 5xi + 5yi^2 - 2x + 2yi = 3+7i$

$x + yi + 5xi - 5y - 2x + 2yi = 3+7i$

$-x - 5y + (3y+5x)i = 3+7i$

Equate real + imag

$-x - 5y = 3$ — (1)

$3y + 5x = 7$ — (2)

3b contd from ① $x = -3 - 5y$ — ③

in ② $3y + 5(-3-5y) = 7$

$3y - 15 - 25y = 7$

$-22y = 22$

$y = -1$

in ③ $x = -3 - 5(-1)$

$x = -3 + 5$

$x = 2$

$\therefore A = 2 + -1i = 2 - i$

$B = 2 - -1i = 2 + i$

④ $(x+iy)(2+i) = 3 - i$

$2x + xi + 2yi + yi^2 = 3 - i$

$2x - y + (x+2y)i = 3 - i$

equate real + imag

$2x - y = 3$ — ①

$x + 2y = -1$ — ②

from ① $y = 2x - 3$ — ③

in ② $x + 2(2x-3) = -1$

$x + 4x - 6 = -1$

$5x = 5$

$x = 1$

in ③ $y = -1$

(7) $$\frac{1}{x+iy} = 2-3i$$

$$x+iy = \frac{1}{2-3i}$$

$$x+iy = \frac{1}{2-3i} \times \frac{2+3i}{2+3i}$$

$$x+iy = \frac{2+3i}{4-9i^2}$$

$$x+iy = \frac{2+3i}{13}$$

$$x+iy = \frac{2}{13} + \frac{3}{13}i \quad \therefore \quad x = \frac{2}{13},\ y = \frac{3}{13}$$

(6) $$(x+iy)(3+4i) = 3-4i$$

$$3x + 4xi + 3yi + 4yi^2 = 3-4i$$

$$3x - 4y + (4x+3y)i = 3-4i$$

Compare real & imag

$$3x - 4y = 3 \quad — (1)$$

$$4x + 3y = -4 \quad — (2)$$

(1)×4 $$12x - 16y = 12 \quad — (3)$$

(2)×3 $$12x + 9y = -12 \quad — (4)$$

(3) − (4) $$-25y = 24$$

$$y = -\frac{24}{25}$$

in (1) $$3x - 4\left(-\frac{24}{25}\right) = 3$$

$$3x + \frac{96}{25} = 3$$

$$3x = \frac{-21}{25}$$

$$x = \frac{-7}{25}$$

(9) $(a-bi)^2 = -4$

$a^2 - 2abi + b^2i^2 = -4$

$(a^2-b^2) - 2abi = -4$

Comp real + imag

$a^2 - b^2 = -4$ $\qquad -2ab = 0$

$\cancel{a^2 - b^2 + 4 = 0}$ $\qquad a = 0$

$\therefore 0^2 - b^2 = -4$

$b^2 = 4$

$b = \pm 2$

if $b = 0$ $\quad a^2 = -4$

$a = \pm\sqrt{4}$ but $a \in \mathbb{R}$

$\therefore a = 0$ and $b = \pm 2$

(10) $Z = 5 - 12i$

$$\frac{1}{Z} = \frac{1}{5-12i} \times \frac{5+12i}{5+12i} = \frac{5+12i}{25-144i^2} = \frac{1}{169}(5+12i)$$

Now if $a+ib = z^{\frac{1}{2}}$

then $(a+ib)^2 = 5-12i$

$a^2 + 2abi + i^2b^2 = 5 - 12i$

Comp real + imag

$a^2 - b^2 = 5$ — (1)

$2ab = -12$ — (2)

from (2) $b = -\frac{6}{a}$

in (1) $a^2 - \left(\frac{-6}{a}\right)^2 = 5$

$a^2 - \frac{36}{a^2} = 5$

$a^4 - 36 = 5a^2$

$a^4 - 5a^2 - 36 = 0$

$(a^2-9)(a^2+4) = 0$

$\therefore$ either $a^2 = 9$ or $a^2 + 4 = 0$

$a = \pm 3$ $\qquad a^2 \neq -4$

$a \in \mathbb{R}$

when $a = +3$ $\quad b = -2$

$a = -3$ $\quad b = +2$

$\therefore z^{\frac{1}{2}} = \pm(3 \mp 2i)$

(11) $$\frac{x}{1+i} - \frac{y}{2-i} = \frac{1-5i}{3-2i}$$

$$\frac{x}{1+i} - \frac{y}{2-i} - \frac{1-5i}{3-2i} = 0$$

$$\frac{x(2-i)(3-2i) - y(1+i)(3-2i) - (1-5i)(1+i)(2-i)}{(1+i)(2-i)(3-2i)} = 0$$

$$x(6-4i-3i+2i^2) - y(3-2i+3i-2i^2) - (1-5i)(2-i+2i-i^2) = 0$$

$$x(4-7i) - y(5+i) - (1-5i)(3+i) = 0$$

$$4x - 7xi - 5y - yi - (3+i-15i-5i^2) = 0$$

$$4x - 7xi - 5y - yi - 8 + 14i = 0$$

$$(4x - 5y - 8) + (-7x - y + 14) = 0$$

$$\therefore \quad 4x - 5y = 8 \quad -(1)$$
$$7x + y = 14 \quad -(2)$$

From (2) $y = 14 - 7x \quad -(3)$

in (1) $4x - 5(14-7x) = 8$

$$4x - 70 + 35x = 8$$
$$39x = 78$$
$$x = 2$$

in (3) $y = 14 - 7(2) = 0.$

(12) $z_1 = 2-3i \qquad z_2 = 5+4i$

(a) $\dfrac{z_1 z_2}{z_1 + z_2}$

$z_1 z_2 = 10 + 8i - 15i - 12i^2 = 22 - 7i$

$z_1 + z_2 = 7 + i$

$\therefore \dfrac{z_1 z_2}{z_1 + z_2} = \dfrac{22-7i}{7+i} \times \dfrac{7-i}{7-i} = \dfrac{154 - 22i - 49i + 7i^2}{49 - i^2}$

$= \dfrac{147 - 71i}{50}$

(b) $m(2-3i) + n(5+4i) = 11 + 18i$

$2m - 3mi + 5n + 4ni = 11 + 18i$

Comp real & imag

$2m + 5n = 11 \quad - (1)$

$-3m + 4n = 18 \quad - (2)$

$(1) \times 3 \quad 6m + 15n = 33 \quad +$

$(2) \times 2 \quad -6m + 8n = 36$

$23n = 69$

$n = 3$

in (1) $2m + 15 = 11$

$2m = -4$

$m = -2$

(8) $\dfrac{1}{x+iy} + \dfrac{1}{1+2i} = 1$

$\dfrac{1+2i + x+iy}{(x+iy)(1+2i)} = 1$

$(1+x)+(2+y)i = (x+iy)(1+2i)$

$(1+x)+(2+y)i = x + 2xi + yi + 2yi^2$

$(1+x)+(2+y)i = (x-2y) + (2x+y)i$

Compare real & imag

$(1+x) = x-2y$

$y = \frac{-1}{2}$

$2+y = 2x+y$

$x = 1.$

(13) $(2-i)x - (1+3i)y - 7 = 0$

$2x - xi - y - 3yi = 7$

$2x-y + (-x-3y)i = 7$

Compare real & imag

$2x - y = 7$ — (1)

$-x-3y = 0$ — (2)

$x = -3y$ — (3)

in (1) $2(-3y) - y = 7$

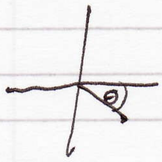
$-6y - y = 7$

$-7y = 7$

$y = -1$

in (3) $x = -3(-1) = 3$

$\therefore z = 3 - i$

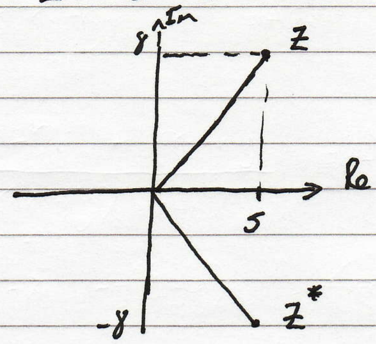


(a) $|z| = \sqrt{3^2+(-1)^2} = \sqrt{10}$

(b) $\arg z = \tan^{-1}\left(\frac{-1}{3}\right) = -18.4^\circ$

(14) (a) $z = 5 + 8i \qquad z^* = 5 - 8i$

(i)



(ii) $z + 2z^* = 5 + 8i + 2(5 - 8i)$

$= 5 + 8i + 10 - 16i$

$= 15 - 8i$

$|z + 2z^*| = \sqrt{15^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289} = 17$

(iii) $\operatorname{Arg}(z + 2z^*) = \tan^{-1}\left(\frac{-8}{15}\right) = -0.49^c$

(b) let $w = m + ni \qquad w^* = m - ni$

$(m + ni)(m - ni) = 5$

$m^2 - mni + mni - n^2 i^2 = 5$

$m^2 + n^2 = 5 \quad — \quad (1)$

$\frac{m + ni}{m - ni} = \frac{1}{5}(-3 + 4i)$

~~[illegible]~~ $\frac{(m + ni)}{(m - ni)} \times \frac{(m + ni)}{(m + ni)} = \frac{1}{5}(-3 + 4i)$

$\frac{m^2 + mni + mni + n^2 i^2}{m^2 + mni - mni - n^2 i^2} = \frac{1}{5}(-3 + 4i)$

$\frac{m^2 - n^2 + 2mni}{m^2 + n^2} = \frac{1}{5}(-3 + 4i)$

14(b) Contd using

but from ①

$$\frac{m^2-n^2+2mni}{\cancel{5}} = \frac{1}{\cancel{5}}(-3+4i)$$

Comparing real & imag.

$$m^2-n^2=-3 \quad \text{—②}$$

$$2mn=4 \quad \text{—③}$$

from ③ $m=\frac{2}{n}$ —④

in ② $\left(\frac{2}{n}\right)^2-n^2=-3$

$$\frac{4}{n^2}-n^2=-3$$

$$4-n^4=-3n^2$$

$$n^4-3n^2-4=0$$

$$(n^2-4)(n^2+1)=0$$

$\therefore$ either $n^2=4$ or $n^2=-1$ × $n\in\mathbb{R}$

$n=\pm 2$

in ④ $m=\pm 1$

$\therefore$ $w=1+2i$ or $w=-1-2i$

(15) $(5+6i)(3-2i) = A+Bi$

$15 - 10i + 18i - 12i^2 = A+Bi$

$27 + 8i = A+Bi$

$\therefore \quad A = 27 \quad B = 8$

$(5-6i)(3+2i) = 15 + 10i - 18i - 12i^2 = 27 - 8i$

ie if $z_1 z_2 = z_3$

then $z_1^* z_2^* = z_3^*$

Now $(27-8i)(27+8i) = (5+6i)(3-2i)(5-6i)(3+2i)$

$27^2 + 216i - 216i - 8^2 i^2 = (5+6i)(5-6i)(3-2i)(3+2i)$

$27^2 + 8^2 = (5^2 - 6^2 i^2)(3^2 - 2^2 i^2)$

$27^2 + 8^2 = (5^2 + 6^2)(3^2 + 2^2)$

(16) $z^4 - 1 = 0$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

let $z = a+ib \qquad (a+ib)^4 = 1$

$a^4 + 4a^3(ib)^1 + 6a^2(ib)^2 + 4a(ib)^3 + (ib)^4 = 1$

$a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4 = 1.$

Comp real + imag

$a^4 - 6a^2b^2 + b^4 = 1$

$4a^3b - 4ab^3 = 0$

$a^2 - b^2 = 0 \qquad (a+b)(a-b) = 0$

$a^2 = b^2 \qquad a = \pm b$

$b^4 - 6b^4 + b^4 = 1.$

$-4b^4 = 1$

$b^4 = \frac{1}{4}$

(16) $z^4 - 1 = 0$

$(z^2+1)(z^2-1) = 0$

$\therefore$ either $z^2+1=0$ or $z^2-1=0$

$z^2 = -1 \qquad z^2 = 1$

$z = \pm i \qquad z = \pm 1$

(17) $2z^3 - 9z^2 + 30z - 13 = 0$

if $2+3i$ is a root then so is $2-3i$

$\therefore$ $z-(2+3i) = z-2-3i$ is a factor

and $z-(2-3i) = z-2+3i$ is a factor.

$\therefore$ $(z-2-3i)(z-2+3i) = x^2 - 2x + 3xi - 2x + 4 - 6i - 3xi + 6i - 9i^2$

$= z^2 - 4z + 13$ is a factor

Now

$$
\begin{array}{r|l}
 & 2z - 1 \\
z^2 - 4z + 13 & 2z^3 - 9z^2 + 30z - 13 \\
 & 2z^3 - 8z^2 + 26z \\
 & \quad -z^2 + 4z - 13 \\
 & \quad -z^2 + 4z - 13 \\
 & \quad \cdot \quad \cdot \quad \cdot
\end{array}
$$

$\therefore$ $2z - 1 = 0$

$z = \frac{1}{2}$ is the third root

(18) $2z^3 - 5z^2 + 12z - 5 = 0$

$1-2i$ is a root then so is $1+2i$

$\therefore$ $z-(1-2i) = z-1+2i$ is a factor

and $z-(1+2i) = z-1-2i$ is a factor

$\therefore$ $(z-1+2i)(z-1-2i) = z^2 - z - \cancel{2zi} - z + 1 + \cancel{2i} + \cancel{2zi} - \cancel{2i} - 4i^2$

$= z^2 - 2z + 5$ is a factor

Now

$$
\begin{array}{r|l}
 & 2z - 1 \\
z^2 - 2z + 5 & 2z^3 - 5z^2 + 12z - 5 \\
 & 2z^3 - 4z^2 + 10z \\
 & \quad -z^2 + 2z - 5 \\
 & \quad -z^2 + 2z - 5 \\
 & \quad \cdot \quad \cdot \quad \cdot
\end{array}
$$

$\therefore$ $2z - 1 = 0$

$z = \frac{1}{2}$ is third factor

(19) $z^4+3z^3+12z-16=0$

$\pm 2i$ are roots

$\therefore$ $z-2i$ and $z+2i$ are factors

$(z-2i)(z+2i) = z^2-4i^2 = z^2+4$

$$\begin{array}{r}
z^2+3z-4 \\
z^2+4 \overline{\big) z^4+3z^3+0z^2+12z-16} \\
z^4 \quad\quad +4z^2 \quad\quad\quad \\
\hline
3z^3-4z^2+12z \quad\quad \\
3z^3 \quad\quad +12z \quad\quad \\
\hline
-4z^2 \quad -16 \\
-4z^2 \quad -16 \\
\hline
\cdot \quad\quad \cdot
\end{array}$$

$\therefore$ $(z^2+3z-4)(z-2i)(z+2i)=0$

$(z+4)(z-1)(z-2i)(z+2i)=0$

$\therefore$ $z = \pm 2i$ or 1 or -4

(20) $z^4+5z^2+4=0$

$(z^2+1)(z^2+4)=0$

either $z^2+1=0$ or $z^2+4=0$

$z^2=-1$ $\quad$ $z^2=-4$

$z=\pm i$ $\quad$ $z=\pm 2i$

(21) $f(z) \equiv z^3-3z^2+4z-4$

If z^2-z+2 is a factor then it will leave no remainder when divided in.

$$\begin{array}{r}
z-2 \\
z^2-z+2 \overline{\big) z^3-3z^2+4z-4} \\
z^3-z^2+2z \quad\quad \\
\hline
-2z^2+2z-4 \\
-2z^2+2z-4 \\
\hline
\cdot \quad \cdot \quad \cdot
\end{array}$$

$\leftarrow$ no remainder $\therefore$ factor

$\therefore f(z) \equiv (z^2-z+2)(z-2)$

either $z-2=0$ or $z^2-z+2=0$

$z=2$ $\quad$ $(z-\frac{1}{2})^2+\frac{7}{4}=0$

$(z-\frac{1}{2})^2 = -\frac{7}{4}$

$z-\frac{1}{2} = \pm\frac{\sqrt{7}i}{2}$

$z=\frac{1}{2}(1\pm\sqrt{7}i)$ or $z=2$.

(22) $z^3 - bz^2 + a^2z - a^2b = 0$

if ai is a root

then $(ai)^3 - b(ai)^2 + a^2(ai) - a^2b = 0$

$-a^3i + \cancel{a^2b} + a^3i - \cancel{a^2b} = 0$

$0 = 0$

$\therefore$ ai is a root, as is $-ai$

$\therefore$ $z - ai$ is a factor as is $z + ai$

hence $(z - ai)(z + ai) = z^2 + a^2$

$$
\begin{array}{r|l}
 & z \quad - b \\
z^2 + a^2 & z^3 - bz^2 + a^2z - a^2b \\
 & z^3 \qquad\quad + a^2z \\
\hline
 & \quad -bz^2 \qquad\quad - a^2b \\
 & \quad -bz^2 \qquad\quad - a^2b \\
\hline
 & \quad - \qquad\qquad\quad \cdot
\end{array}
$$

$\therefore$ $z - b = 0$

$z = b$ is the third root