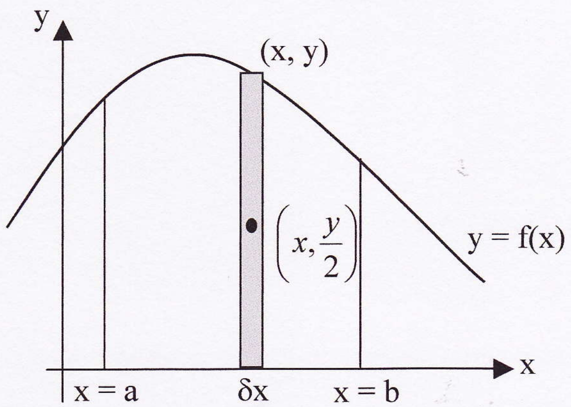
<u>Calculus methods for determining the centre of mass of a uniform plane lamina</u>

In M2 we considered how to determine the position of the centre of mass of a uniform plane lamina using its symmetries.

$$\sum m_i \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \sum m_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

We must now consider how to deal with lamina which have no symmetries.

When a lamina or plane region is bounded by curves for which the equations are known, calculus methods can be used to find the position of its centre of mass.



When a lamina occupies a region between a curve and the x-axis as shown above, it can be approximated by elementary strips of width δx and length y. One such strip, situated at the point (x, y) on the curve is shown. The dot indicates the centre of mass of this elementary strip. Let ρ (Greek letter 'rho') be the mass per unit area of the lamina.

Mass of each strip, δM = AREA of STRIP × DENSITY = $y\delta x\rho$

Centre of mass of strip @ $\left(x, \frac{y}{2}\right)$

Hence centre of mass of lamina $(\bar{x}, \bar{y})$ can be determined:

$(\sum \delta M)\bar{x} = \sum(\delta M x)$

$(\sum \cancel{\rho} y\delta x)\bar{x} = \sum(\cancel{\rho} y\delta x\, x)$

$(\sum y\delta x)\bar{x} = \sum(xy\delta x)$

in limit as $\delta x \to 0$, summations become integrals

$\left(\int_a^b y\,dx\right)\bar{x} = \int_a^b xy\,dx$

$\Downarrow$

$(\sum \delta M)\bar{y} = \sum\left(\delta M\left(\frac{y}{2}\right)\right)$

$(\sum y\delta x\cancel{\rho})\bar{y} = \sum\left(y\delta x\cancel{\rho}\left(\frac{y}{2}\right)\right)$

$(\sum y\delta x)\bar{y} = \frac{1}{2}\sum y^2\delta x$

in limit as $\delta x \to 0$

$\left(\int_a^b y\,dx\right)\bar{y} = \frac{1}{2}\int_a^b y^2\,dx$

$\Downarrow$

In summary:

$$M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\int_a^b \rho xy\,dx\\ \frac{1}{2}\int_a^b \rho y^2\,dx\end{pmatrix}$$

where the total mass, $M = \int_a^b \rho y\,dx$

Examples

1. Find the position of the centre of mass of a uniform semi-circular lamina of radius r.

2. Find the coordinates of the centre of mass of the region between the curve $y = 4 - x^2$ and the positive x and y axis.

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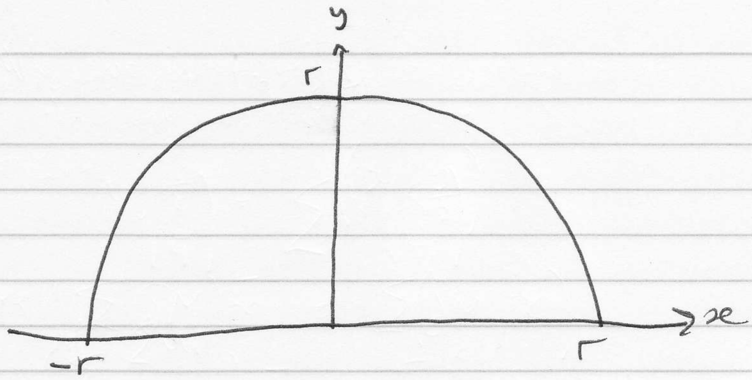
Eg1



$$x^2 + y^2 = r^2$$

By symmetry, $\bar{x} = 0$

Mass of semi-circle $= \frac{1}{2}\rho\pi r^2$

Now $\quad M\bar{y} = \frac{1}{2}\int_{-r}^{r} \rho y^2 \, dx$

where $\quad y^2 = r^2 - x^2$

$$\cancel{\tfrac{1}{2}}\cancel{\rho}\pi r^2 \bar{y} = \cancel{\tfrac{1}{2}}\cancel{\rho}\int_{-r}^{r} r^2 - x^2 \, dx$$

$$\pi r^2 \bar{y} = \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$\pi r^2 \bar{y} = \left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 - \frac{-r^3}{3} \right)$$

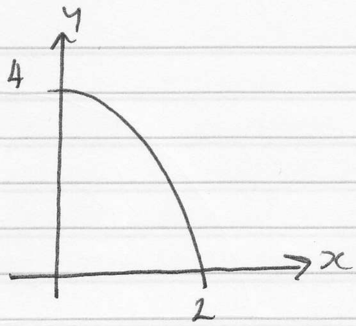
$$\pi r^2 \bar{y} = r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}$$

$$\pi r^2 \bar{y} = \frac{4r^3}{3}$$

$$\bar{y} = \frac{4r}{3\pi}$$ $\Leftarrow$ standard result established in M2.

Eg2 $y = 4 - x^2$

$x=0 \quad y=4$
$y=0 \quad x=\pm 2$



$$M = \rho \int_0^2 y \, dx$$

$$M = \rho \int_0^2 4 - x^2 \, dx$$

$$M = \rho \left[4x - \frac{x^3}{3} \right]_0^2$$

$$M = \rho \left(8 - \frac{8}{3} \right)$$

$$M = \frac{16\rho}{3}$$

Now $M\bar{x} = \int \rho x y \, dx$

$$\frac{16\cancel{\rho}}{3} \bar{x} = \cancel{\rho} \int_0^2 x(4-x^2) \, dx$$

$$\frac{16}{3} \bar{x} = \left[2x^2 - \frac{x^4}{4} \right]_0^2$$

$$\frac{16}{3} \bar{x} = \left(8 - \frac{16}{4} \right)$$

$$\frac{16}{3} \bar{x} = 4$$

$$\bar{x} = \frac{3}{4}$$

AND $M\bar{y} = \frac{1}{2} \int \rho y^2 \, dx$

$$\frac{16\cancel{\rho}}{3} \bar{y} = \frac{1}{2} \cancel{\rho} \int_0^2 (4-x^2)^2 \, dx$$

$$\frac{32}{3} \bar{y} = \int_0^2 16 - 8x^2 + x^4 \, dx$$

$$\frac{32}{3} \bar{y} = \left[16x - \frac{8}{3} x^3 + \frac{x^5}{5} \right]_0^2$$

$$\frac{32}{3} \bar{y} = 32 - \frac{64}{3} + \frac{32}{5}$$

$$\frac{32}{3} \bar{y} = \frac{256}{15}$$

$$\bar{y} = \frac{8}{5}$$

$\therefore$ COM @ $\left(\frac{3}{4}, \frac{8}{5} \right)$