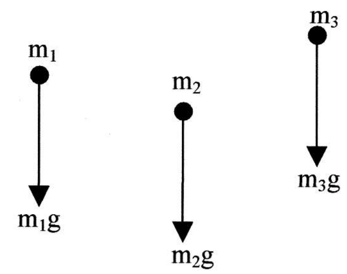
Centre of Mass

Every particle is attracted to the centre of the Earth, and this force of attraction is known as the weight of the particle.

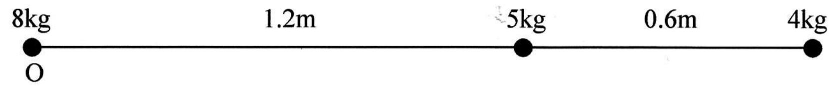
For a number of particles m_1, m_2, m_3, these forces may be considered to be parallel, all being directed towards the centre of the Earth. If the relative positions of these particles are fixed and known, then the resultant of these parallel forces can be determined.



Centre of Mass of a System of Particles

The centre of mass of a number of particles is the point through which the line of action of the resultant of these parallel forces always passes.

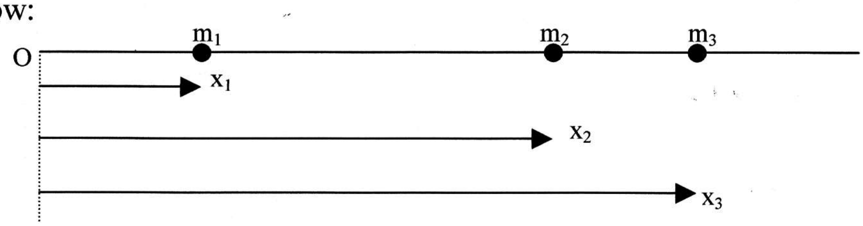
Eg1 An object consists of three point masses 8kg, 5kg and 4kg attatched to a rigid light rod as shown.



Calculate the distance of the centre of mass of the object from end O.

Note that although the force of gravity was included in the calculation, it cancelled out. The answer depends only on the masses and their distances from the origin and not on the value of g. This leads to the following definition for the position of the centre of mass.

Consider a set of n point masses, $m_1, m_2, \ldots, m_n$ attatched to a rigid light rod (whose mass is neglected) at positions $x_1, x_2, \ldots, x_n$ from one end O. The situation is shown below:



The position of the centre of mass from O, labelled $\bar{x}$, is defined by the equation

Moment of whole mass at centre of mass = sum of moments of individual masses

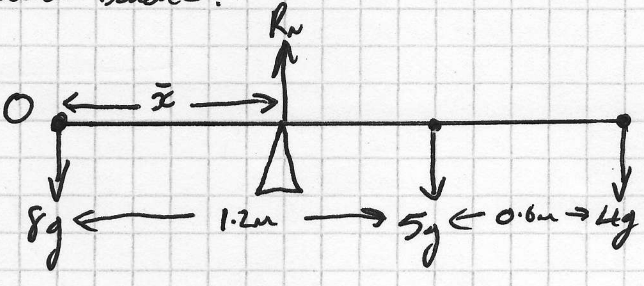
Or
$$M\bar{x} = \sum_{i=1}^{n} m_i x_i$$

Where M is the total mass.

Eg2 A uniform rod of length 2m has mass 5kg. Masses of 4kg and 6kg are fixed at each end of the rod. Find the centre of mass of the rod.

Exercise 2A Pg 37 Q's 1, 2, 4, 5, 6

Eg1 Suppose the centre of mass C is $\bar{x}$ m from O: if a pivot were at this position the rod would balance.



For equilibrium, $\Sigma F_y = 0$ $\quad R_N = 8g + 5g + 4g = 17g$ N.

Also $\Sigma C = 0$

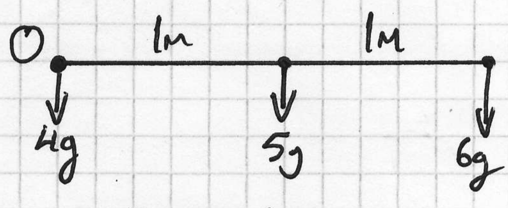
$$C_O: \quad R_N \times \bar{x} + (1.2 \times {}^-5g) + (1.8 \times {}^-4g) = 0$$

$$17g\bar{x} = 6g + 7.2g$$

$$\bar{x} = \frac{13.2}{17} = 0.776$$

∴ the COM is 0.776 m from the end O of the rod.

Eg2



$$M\bar{x} = \Sigma M_i x_i$$

$$(4+5+6)\bar{x} = (4\times0) + (5\times1) + (6\times2)$$

$$15\bar{x} = 5 + 12$$

$$\bar{x} = \frac{17}{15} = 1.1\dot{3}$$

∴ COM is $1.1\dot{3}$ m from the 4kg point mass.

Centres of mass for 2-dimensional bodies

The techniques developed for finding the centre of mass using moments can be extended into two dimensions.

If a two dimensional body consists of a set of n point masses m_1, m_2, …,m_n located at positions (x_1,y_1), (x_2, y_2), …, (x_n,y_n) then the position of the centre of mass of the body $(\bar{x}, \bar{y})$ is given by

$$M\bar{x} = \sum_{i=1}^{n} m_i x_i \qquad \text{and} \qquad M\bar{y} = \sum_{i=1}^{n} m_i y_i$$

These can be combined into a two-dimensional vector format, which speeds up calculations:

$$M\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \sum_{i=1}^{n} m_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

<u>Eg3</u> Particles A, B and C of mass 10kg, 15kg and 25kg are situated at points with co-ordinates (2, 3), (4,2) and (6,6) respectively. Find the co-ordinates of the centre of mass of the system.

<u>Eg4</u> A light rectangular plate ABCD where AB = 20cm and AD = 30cm has particles of masses 4kg, 4kg, 5kg and 2kg attached at A, B, C and D respectively. Find the position of the centre of mass of the resulting system.

<u>Exercise 2A</u> Q's 3, 7 to 13.

Eg3

$$M\bar{x} = \sum M_i x_i$$

$$50\bar{x} = (10\times2)+(15\times4)+(25\times6)$$

$$50\bar{x} = 20 + 60 + 150$$

$$\bar{x} = \frac{230}{50} = 4.6$$

$$M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \sum M\begin{pmatrix}x_i\\ y_i\end{pmatrix}$$

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 10\begin{pmatrix}2\\3\end{pmatrix}+15\begin{pmatrix}4\\2\end{pmatrix}+25\begin{pmatrix}6\\6\end{pmatrix}$$

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}20\\30\end{pmatrix}+\begin{pmatrix}60\\30\end{pmatrix}+\begin{pmatrix}150\\150\end{pmatrix}$$

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}230\\210\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}4.6\\4.2\end{pmatrix}$$

$$M\bar{y} = \sum M_i y_i$$

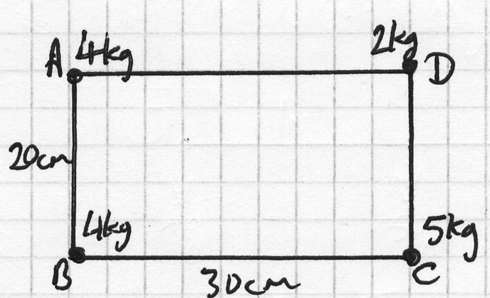
$$50\bar{y} = (10\times3)+(15\times2)+(25\times6)$$

$$50\bar{y} = 30 + 30 + 150$$

$$\bar{y} = \frac{210}{50} = 4.2$$

$\therefore$ COM @ $(4.6, 4.2)$

Eg4



Taking B as origin:

$$M\bar{x} = \sum M_i x_i$$

$$15\bar{x} = (4\times0)+(4\times0)+(5\times30)+(2\times30)$$

$$15\bar{x} = 150+60$$

$$\bar{x} = \frac{210}{15} = 14\text{cm}$$

$$M\bar{y} = \sum M_i y_i$$

$$15\bar{y} = (4\times20)+(4\times0)+(5\times0)+(2\times20)$$

$$15\bar{y} = 80+40$$

$$\bar{y} = \frac{120}{15} = 8\text{cm}.$$

$\therefore$ COM @ $(14, 8)$ relative to B.

Centre of Mass of a Uniform Plane Lamina

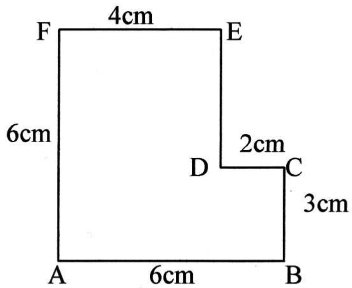
M2 Pgs 39&40.

Exercise 2B Pg 47 Q1, 3 & 4

Composite Bodies

By considering the masses, and the positions of the centres of mass of each constituent part of a composite body, its centre of mass can be calculated.

Eg1 (a) The diagram shows a uniform lamina ABCDEF. By taking the corner A as the origin, and assuming that each part has mass per unit area m kg, calculate the position of the centre of mass of the lamina.



(b) A circle of radius 1cm, centred at (2,4) is now cut out of the lamina. Calculate the new position of the centre of mass.

Exercise 2B Q2

Centre of Mass of a Framework

Eg2 A uniform wire of length 30cm is bent to form a triangle ABC where AB has length 5cm and AC is of length 12cm. By taking A as the origin, calculate the position of the centre of mass of the framework.

Exercise 2B Q's 5 to 15 Odd's

Equilibrium of a Suspended Body

When a body is suspended, the centre of mass lies immediately below the point of suspension.

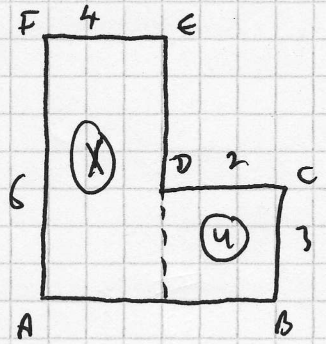
Eg3 The lamina in Eg 1 is now freely suspended by string from the corner A. Calculate the angle which the side AB makes to the vertical when the lamina is in equilibrium.

Equilibrium of Lamina on an inclined plane

Eg4 Consider the lamina from Eg1 resting on an adjustable inclined plane, with AB in contact, and side AF nearest the base. What is the maximum possible angle of inclination of the plane?

Exercise 2C Primes

Eg 1 (a)



Area of $(X) = 6 \times 4 = 24\text{cm}^2$

$\therefore$ Mass of $(X) = 24m$ kg

Area of $(Y) = 2 \times 3 = 6\text{cm}^2$

$\therefore$ Mass of $(Y) = 6m$ kg

Now $\quad M\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \sum m_i\begin{pmatrix}x_i\\y_i\end{pmatrix}$

$$(24\mu + 6\mu)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = 24\mu\begin{pmatrix}2\\3\end{pmatrix} + 6\mu\begin{pmatrix}5\\1.5\end{pmatrix}$$

$$30\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}48\\72\end{pmatrix} + \begin{pmatrix}30\\9\end{pmatrix}$$

$$30\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}78\\81\end{pmatrix}$$

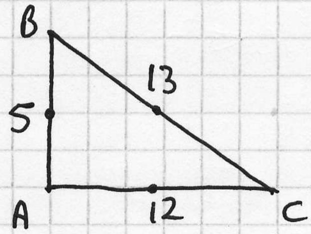
$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}2.6\\2.7\end{pmatrix}$$

(b) $$(24\mu + 6m - \pi 1^2\mu)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = 24\mu\begin{pmatrix}2\\3\end{pmatrix} + 6\mu\begin{pmatrix}5\\1.5\end{pmatrix} - \pi\mu\begin{pmatrix}2\\4\end{pmatrix}$$

$$(30 - \pi)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}78\\81\end{pmatrix} - \begin{pmatrix}2\pi\\4\pi\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}2.67\\2.55\end{pmatrix}$$

Eg 2



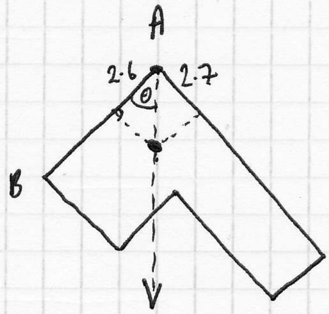
$$(5 + 12 + 13)\mu\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = 5\mu\begin{pmatrix}0\\2.5\end{pmatrix} + 12\mu\begin{pmatrix}6\\0\end{pmatrix} + 13\mu\begin{pmatrix}6\\2.5\end{pmatrix}$$

$$30\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}0\\12.5\end{pmatrix} + \begin{pmatrix}72\\0\end{pmatrix} + \begin{pmatrix}78\\32.5\end{pmatrix}$$

$$30\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}150\\45\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}5\\1.5\end{pmatrix}$$

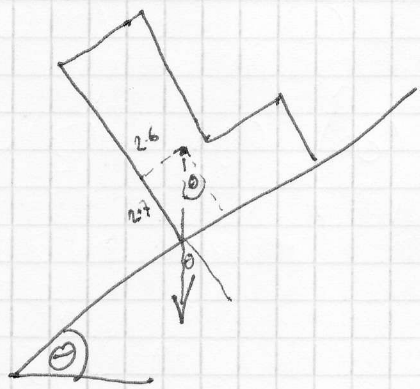
Eg3

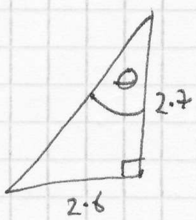


$$\tan\theta = \frac{2.7}{2.6}$$

$$\theta = 46.1^{\circ}$$

What ~~ops~~ about toppling on an inclined plane?





$$\tan\theta = \left(\frac{2.6}{2.7}\right) = 44^{\circ}$$