Centre of Mass of a Uniform Solid Body

M3 Textbook Pages 172 – 174

Exercise 5B Pg 183 Q's 1 to 4

Centres of Mass of Solids of Revolution

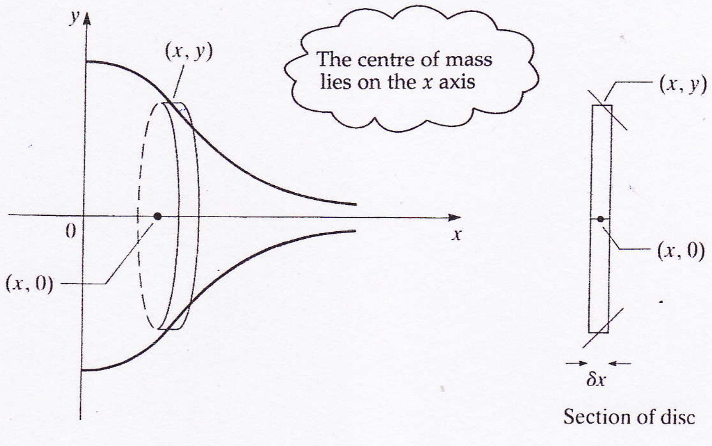
In C4 you used calculus methods to determine the volumes of solids of revolution, i.e. if the curve y = f(x) is rotated completely about the x axis then the volume of the solid generated between x = a and x = b is given by:

$$V = \pi \int_a^b y^2 \, dx$$

This can now be extended to find their centres of mass (assuming they are of uniform density).

Notice that because of symmetry, the centre of mass of a solid of revolution must lie on its axis, so provided the x or y axis becomes the axis of symmetry, there is only one coordinate to determine.

Take a solid of revolution about the x axis and divide the solid into thin discs. For an elementary disc situated at the point (x, y) on the curve, the centre of mass is at the point (x, 0) on the x axis as shown below:



The volume of the disc is: $\delta V = \pi y^2 \delta x$

and if the density is ρ, its mass is: $\delta M = \rho \pi y^2 \delta x$

The solid is now approximated by the sum of all such discs. For each of these both its mass and the position of its centre of mass are known, so for the composite body, the position $\bar{x}$ of the centre of mass is given by:

$$M\bar{x} = \sum_i m_i x_i$$

In our case:

$$\left(\sum_{\substack{\text{all}\\\text{discs}}} \delta M\right)\bar{x} = \sum_{\substack{\text{all}\\\text{discs}}}(\delta M x)$$

$$\left(\sum \rho\pi y^2 \delta x\right)\bar{x} = \sum\left(\rho\pi y^2 x\, \delta x\right)$$

In limit as $\delta x \to 0$ $\sum \Rightarrow \int$

$$\left(\int \rho\pi y^2 dx\right)\bar{x} = \int \rho\pi y^2 x\, dx$$

$$\bar{x} = \frac{\rho\pi \int y^2 x\, dx}{\rho\pi \int y^2\, dx} = \frac{\int y^2 x\, dx}{\int y^2\, dx}$$

But more usefully: $M\bar{x} = \rho\pi \int y^2 x\, dx$

where $M = \rho\pi \int y^2 dx$

Examples

1. An area is enclosed by the curve $y^2 = 5x$, the x axis the lines $x = 1$, $x = 3$ and it lies in the first quadrant. The area is rotated through one revolution. Find the coordinates of the centre of mass of the uniform solid so formed.

2. Find the position of the centre of mass of the solid of revolution formed when the region between the curve $y = 2e^x$, the lines $x = 0$, $x = 2$ and the x axis is rotated through 360^o about the x axis.

3. Use integration to show that the centre of mass of a uniform solid hemisphere of radius r, is at a distance 3r/8 from the centre C of the plane circular base.

Exercise 5B Pg 183, Q's 5 to 11 Odds

Then Q's 12 to 16 together

Eg1 $\quad M\bar{x} = \rho\pi \int y^2 x \, dx$

where $M = \rho\pi \int y^2 \, dx$

Now $y^2 = 5x \quad \therefore \quad M = \rho\pi \int_1^3 5x \, dx$

$$M = \rho\pi \left[\frac{5x^2}{2} \right]_1^3 = \rho\pi \left(\frac{45}{2} - \frac{5}{2} \right) = 20\rho\pi$$

Now $20\cancel{\rho\pi}\,\bar{x} = \cancel{\rho\pi} \int_1^3 (5x)\,x \, dx$

$$20\bar{x} = \int_1^3 5x^2 \, dx$$

$$20\bar{x} = \left[\frac{5x^3}{3} \right]_1^3$$

$$20\bar{x} = \left(\frac{135}{3} - \frac{5}{3} \right)$$

$$20\bar{x} = \frac{130}{3}$$

$$\bar{x} = \frac{13}{6}$$

Eg 2 $M = \rho\pi \int y^2\,dx$

$$M = \rho\pi \int_0^2 (2e^x)^2\,dx$$

$$M = \rho\pi \int_0^2 4e^{2x}\,dx$$

$$M = \rho\pi \left[2e^{2x}\right]_0^2$$

$$M = \rho\pi(2e^4 - 2) = 2\rho\pi(e^4 - 1)$$

$$M\bar{x} = \rho\pi \int y^2 x\,dx$$

$$M\bar{x} = \rho\pi \int_0^2 x \cdot 4e^{2x}\,dx$$

INTEGRATE BY PARTS

$$M\bar{x} = 4\rho\pi \int_0^2 xe^{2x}\,dx$$

let $u = x$ $\quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{2}e^{2x}$

$$M\bar{x} = 4\rho\pi\left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int_0^2 e^{2x}\,dx\right]_0^2$$

$$M\bar{x} = 2\rho\pi\left[xe^{2x} - \frac{1}{2}e^{2x}\right]_0^2$$

$$M\bar{x} = 2\rho\pi\left[\left(2e^4 - \frac{1}{2}e^4\right) - \left(0 - \frac{1}{2}\right)\right]$$

$$M\bar{x} = 2\rho\pi\left[\frac{3}{2}e^4 + \frac{1}{2}\right]$$

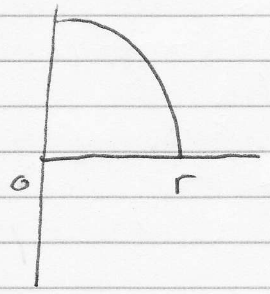
$$M\bar{x} = \rho\pi(3e^4 + 1)$$

Now $2\cancel{\rho\pi}(e^4 - 1)\bar{x} = \cancel{\rho\pi}(3e^4 + 1)$

$$\bar{x} = \frac{(3e^4 + 1)}{2(e^4 - 1)} = 1.54 \text{ to 2dp}$$

$\therefore$ CoM @ $(1.54, 0)$

Eg3 Solid hemisphere = quarter of circle $\times 360°$



$$x^2+y^2=r^2$$
$$y^2=r^2-x^2$$

$$\text{Mass} = \text{Volume} \times \rho = \frac{2}{3}\pi r^3 \rho$$

$$M\bar{x} = \rho\pi \int_0^r y^2 x \, dx$$

$$M\bar{x} = \rho\pi \int_0^r (r^2 - x^2)x \, dx$$

$$M\bar{x} = \rho\pi \int_0^r r^2 x - x^3 \, dx$$

$$M\bar{x} = \rho\pi \left[\frac{r^2x^2}{2} - \frac{x^4}{4} \right]_0^r$$

$$M\bar{x} = \rho\pi \left(\frac{r^4}{2} - \frac{r^4}{4} \right)$$

$$M\bar{x} = \frac{\rho\pi r^4}{4}$$

Now $\frac{2}{3}\pi r^3 \rho \bar{x} = \frac{\rho\pi r^4}{4}$

$$\bar{x} = \frac{3\cancel{\rho\pi} r^{\cancel{4}}}{8\cancel{\pi r^3 \rho}}$$

$$\bar{x} = \frac{3}{8}r \quad \text{As required.}$$