Formation of Second Order Linear Differential Equations

If an equation $y = f(x)$ contains two arbitrary constants, A and B, then by differentiating twice two more equations are produced:

$$\frac{dy}{dx} = f'(x) \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x)$$

These two equations, together with the original $y = f(x)$, allow A and B to be eliminated, so forming a second order differential equation. We will consider three types of equation $y = f(x)$, all of which give rise in this way to a second order linear differential equation.

Case (a)

Consider $y = Ae^{2x} + Be^{3x}$ — (1)

$$\frac{dy}{dx} = 2Ae^{2x} + 3Be^{3x} \quad \text{— (2)}$$

from (1) $Ae^{2x} = y - Be^{3x}$

in (2) $\frac{dy}{dx} = 2(y - Be^{3x}) + 3Be^{3x}$

$$\frac{dy}{dx} = 2y + Be^{3x} \quad \text{— (3)}$$

Now $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 3Be^{3x}$ — (4)

from (3) $Be^{3x} = \frac{dy}{dx} - 2y$

in (4) $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 3\left(\frac{dy}{dx} - 2y\right)$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

or $\frac{d^2y}{dx^2} - (2+3)\frac{dy}{dx} + (2\times3)y = 0$

Compares to quadratic equation

$$m^2 - 5m + 6 = 0$$

So it would appear that the general solution of the second order diff eqn $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ can be found from the roots of the quad eqn $m^2 - 5m + 6 = 0$
$(m-2)(m-3) = 0$
$y = Ae^{2x} + Be^{3x}$

In order to check that this analogy is general, and not just coincidence , we now consider the more general equation

$$y = Ae^{\alpha x} + Be^{\beta x} \quad \text{—}(1)$$

$$\frac{dy}{dx} = \alpha Ae^{\alpha x} + \beta Be^{\beta x} \quad \text{—}(2)$$

From (1) $Ae^{\alpha x} = y - Be^{\beta x}$

In (2) $\frac{dy}{dx} = \alpha(y - Be^{\beta x}) + \beta Be^{\beta x} = \alpha y - (\alpha - \beta)Be^{\beta x}$ —(3)

Now $\frac{d^2y}{dx^2} = \alpha\frac{dy}{dx} - (\alpha - \beta)\beta Be^{\beta x}$ —(4)

From (3) $Be^{\beta x} = \frac{\left(\alpha y - \frac{dy}{dx}\right)}{(\alpha - \beta)}$

In (4) $\frac{d^2y}{dx^2} = \alpha\frac{dy}{dx} - \cancel{(\alpha - \beta)}\beta\frac{\left(\alpha y - \frac{dy}{dx}\right)}{\cancel{(\alpha - \beta)}}$

$\frac{d^2y}{dx^2} = \alpha\frac{dy}{dx} - \alpha\beta y + \beta\frac{dy}{dx}$

$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0$

which compares to the quad eq^n $m^2 - (\alpha + \beta)m + \alpha\beta = 0$ whose roots are α and β.

This is called the *auxiliary quadratic equation* and it can now be used to *recognize* the general solution of a second order linear differential equation with constant coefficients:

For the second order linear differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

If the auxiliary quadratic equation

$$am^2 + bm + c = 0$$

has real distinct roots α and β (ie $b^2 - 4ac > 0$), then we can quote, by recognition, the solution

$$y = Ae^{\alpha x} + Be^{\beta x}$$

eg1 Write down the general solution for each of the following differential equations:

(a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ (b) $\frac{d^2y}{dx^2} - 4y = 0$

Exercise 5A Pg 89 Evens

eg1 (a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Auxiliary Quad eqⁿ $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

either $m = 1$ or $m = 2$

Aux eqⁿ has two distinct real roots

$\therefore$ general solution $y = Ae^{x} + Be^{2x}$

(b) $\frac{d^2y}{dx^2} - 4y = 0$

Aux Quad Eqⁿ $m^2 - 4 = 0$

$m = \pm 2$

$\therefore$ General Solution $y = Ae^{-2x} + Be^{2x}$

Ex 5a

① $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Auxillary eqⁿ $m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

either $m = -2$ or $m = -3$

Aux eqⁿ has two real distinct roots ∴ general solution $y = Ae^{-2x} + Be^{-3x}$

② $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$

A.E. $m^2 - 8m + 12 = 0$

$(m-2)(m-6) = 0$

$m = 2, 6$ ∴ G.S. $y = Ae^{2x} + Be^{6x}$

③ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$

A.E.: $m^2 + 2m - 15 = 0$

$(m+5)(m-3) = 0$

$m = -5, 3$ ∴ G.S. $y = Ae^{-5x} + Be^{3x}$

④ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$

A.E: $m^2 - 3m - 28 = 0$

$(m-7)(m+4) = 0$ ∴ G.S. $y = Ae^{-4x} + Be^{7x}$

⑤ $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$

AE: $m^2 + m - 12 = 0$

$(m+4)(m-3) = 0$ ∴ GS: $y = Ae^{-4x} + Be^{3x}$

(6) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

AE: $m^2 + 5m = 0$

$m(m+5) = 0$

$m = 0, -5$ $\qquad y = Ae^{-5x} + Be^{0} = Ae^{-5x} + B$

(7) $3\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 2y = 0$

AE: $3m^2 + 7m + 2 = 0$

$(3m+1)(m+2) = 0$ $\qquad \therefore GS: y = Ae^{-\frac{1}{3}x} + Be^{-2x}$

(8) $4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$

AE: $4m^2 - 7m - 2 = 0$

$(4m+1)(m-2) = 0$ $\qquad \therefore GS: y = Ae^{-\frac{1}{4}x} + Be^{2x}$

(9) $6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

AE: $6m^2 - m - 2 = 0$

$(3m-2)(2m+1) = 0$ $\qquad GS: y = Ae^{\frac{2}{3}x} + Be^{-\frac{1}{2}x}$

(10) $15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$

AE: $15m^2 - 7m - 2 = 0$

$(3m-2)(5m+1) = 0$ $\qquad y = Ae^{\frac{2}{3}x} + Be^{-\frac{1}{5}x}$

Case (b)

Consider $y = e^{2x}(A + Bx)$ $= Ae^{2x} + Bxe^{2x}$

$$\frac{dy}{dx} = 2Ae^{2x} + Bx.2e^{2x} + e^{2x}.B = 2Ae^{2x} + 2Bxe^{2x} + Be^{2x} = 2y + Be^{2x}$$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} + 2Be^{2x} = 2\frac{dy}{dx} + 2\left(\frac{dy}{dx} - 2y\right)$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Auxiliary quadratic $m^2 - 4m + 4 = 0$ has equal roots of value 2.

Let's generalise: $y = e^{\alpha x}(A + Bx)$

$$\frac{dy}{dx} = e^{\alpha x}.B + (A+Bx)\alpha e^{\alpha x} = Be^{\alpha x} + \alpha Ae^{\alpha x} + \alpha Bxe^{\alpha x}$$
$$= Be^{\alpha x} + \alpha\left[e^{\alpha x}(A+Bx)\right]$$
$$= \alpha y + Be^{\alpha x}$$

$$\frac{d^2y}{dx^2} = \alpha\frac{dy}{dx} + B\alpha e^{\alpha x} = \alpha\frac{dy}{dx} + \alpha\left(\frac{dy}{dx} - \alpha y\right)$$

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + \alpha^2 y = 0$$

$$\frac{d^2y}{dx^2} - (\alpha+\alpha)\frac{dy}{dx} + \alpha\alpha y = 0$$

which compares to quadratic $m^2 - (\alpha+\alpha)m + \alpha\alpha = 0$ whose root is α.

This provides a second form for the solution, by recognition, of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

When the auxiliary quadratic equation $am^2 + bm + c = 0$ has equal roots (ie $b^2 - 4ac = 0$), then

$$y = e^{\alpha x}(A + bx)$$

eg2 Write down the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

Exercise 5B Pg 91 Factors of 10

eg 2 $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

AEQ $m^2 + 6m + 9 = 0$

$$(m+3)(m+3) = 0$$

Auxiliary Quadratic Equation has repeated root $m = -3$

$\therefore$ General Solution $y = e^{-3x}(A + Bx)$.

GX5B

(1) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

AE : $m^2 + 10m + 25 = 0$

$m = -5$ $\quad \therefore$ GS $y = e^{-5x}(A + Bx)$

(2) $\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$

AE : $m^2 - 18m + 81 = 0$

$m = 9$ $\quad \therefore$ GS: $y = e^{9x}(A + Bx)$

(3) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

AE: $m^2 + 2m + 1 = 0$

$m = -1$ $\quad \therefore$ GS: $y = e^{-x}(A + Bx)$

(4) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

AE: $m^2 - 8m + 16 = 0$

$m = +4$ $\quad \therefore$ GS : $y = e^{4x}(A + Bx)$

(5) $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$

AE: $m^2 + 14m + 49 = 0$

$m = -7$ $\quad$ GS: $y = e^{-7x}(A + Bx)$

(6) $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$

AE: $16m^2 + 8m + 1 = 0$

$m = -\frac{1}{4}$ GS: $y = e^{-\frac{1}{4}x}(A+Bx)$

(7) $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

AE: $4m^2 - 4m + 1 = 0$

$m = \frac{1}{2}$ GS: $y = e^{\frac{1}{2}x}(A+Bx)$

(8) $4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$

AE: $4m^2 + 20m + 25 = 0$

$m = -\frac{5}{2}$ GS: $y = e^{-\frac{5}{2}x}(A+Bx)$

(9) $16\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 9y = 0$

AE: $16m^2 - 24m + 9 = 0$

$m = \frac{3}{4}$ $\therefore$ GS: $y = e^{\frac{3}{4}x}(A+Bx)$

(10) $\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$

AE: $m^2 + 2\sqrt{3} + 3 = 0$

$m = -\sqrt{3}$ $\therefore$ GS $y = e^{-\sqrt{3}x}(A+Bx)$

Case (c)

Consider $y = e^{2x}(A\cos 3x + B\sin 3x)$ — ①

diff using product rule:

$$\frac{dy}{dx} = e^{2x}(-3A\sin 3x + 3B\cos 3x) + (A\cos 3x + B\sin 3x)2e^{2x}$$

$$\frac{dy}{dx} = e^{2x}(3B\cos 3x - 3A\sin 3x) + 2y$$

Now

$$\frac{d^2y}{dx^2} = e^{2x}(-9B\sin 3x - 9A\cos 3x) + 2e^{2x}(3B\cos 3x - 3A\sin 3x) + 2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -9e^{2x}(A\cos 3x + B\sin 3x) + 2e^{2x}(3B\cos 3x - 3A\sin 3x) + 2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -9y + 2\left[\frac{dy}{dx} - 2y\right] + 2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Compares to quad eqⁿ $m^2 - 4m + 13 = 0$

$$(m-2)^2 + 9 = 0$$

$$m - 2 = \pm 3i$$

$$m = 2 \pm 3i$$

Compares to ① $y = e^{2x}(A\cos 3x + B\sin 3x)$

Applying the generalising procedure, when the auxiliary quadratic has complex roots $p \pm qi$ we can quote, by recognition the solution

$$y = e^{px}(A\cos qx + B\sin qx)$$

eg3 Write down the general solution for the following differential equations

(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ (b) $\frac{d^2y}{dx^2} + 4y = 0$

From (b) above, we can generalise further that if the auxiliary quadratic has purely imaginary roots, $\pm qi$, then the general solution would be

$$y = A\cos qx + B\sin qx$$

Exercise 5C Pg 92 Evens

Q3 (a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

AQE $m^2 + 2m + 2 = 0$

$(m+1)^2 + 1 = 0$

$(m+1)^2 = -1$

$m+1 = \pm i$

$m = -1 \pm i$

AQE has complex roots $\therefore$ General Solution $y = e^{-x}(A\cos x + B\sin x)$

(b) $\frac{d^2y}{dx^2} + 4y = 0$

AQE $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

AQE has imaginary roots $\therefore$ GS $y = e^{0}(A\cos 2x + B\sin 2x)$

$y = A\cos 2x + B\sin 2x.$

Ex 5C

(1) $\frac{d^2y}{dx^2}+25y=0$

AQE $m^2+25=0$

$m=\pm 5i$ $\therefore$ GS $y=A\cos 5x+B\sin 5x$

(2) $\frac{d^2y}{dx^2}+81y=0$

AQE $m^2+81=0$

$m=\pm 9i$ $\therefore$ GS $y=A\cos 9x+B\sin 9x$

(3) $\frac{d^2y}{dx^2}+y=0$

$m^2+1=0$

$m=\pm i$ $\therefore$ GS $y=A\cos x+B\sin x$

(4) $9\frac{d^2y}{dx^2}+16y=0$

$9m^2+16=0$

$m=\pm \frac{4}{3}i$ $\therefore$ GS $y=A\cos \frac{4}{3}x+B\sin\frac{4}{3}x$

(5) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

$m^2+4m+13=0$

$(m+2)^2+13-4=0$

$(m+2)^2=-9$

$m+2=\pm 3i$

$m=-2\pm 3i$ $\therefore$ GS $y=e^{-2x}(A\cos 3x+B\sin 3x)$

(6) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$

AQE $m^2 + 8m + 17 = 0$

$(m+4)^2 + 1 = 0$

$m + 4 = \pm i$

$m = -4 \pm i$ $\therefore$ GS $y = e^{-4x}(A\cos x + B\sin x)$

(7) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

AQE $m^2 - 4m + 5 = 0$

$(m-2)^2 + 1 = 0$

$m - 2 = \pm i$

$m = 2 \pm i$ $\therefore$ GS $y = e^{2x}(A\cos x + B\sin x)$

(8) ~~$\frac{dy}{dx}$~~ $\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$

AQE $m^2 + 20m + 109 = 0$

$(m+10)^2 + 9 = 0$

$m + 10 = \pm 3i$

$m = -10 \pm 3i$ $\therefore$ GS $y = e^{-10x}(A\cos 3x + B\sin 3x)$

(9) $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$

AQE $9m^2 - 6m + 5 = 0$

$m^2 - \frac{6}{9}m + \frac{5}{9} = 0$

$\left(m - \frac{6}{18}\right)^2 + \frac{5}{9} - \frac{36}{324} = 0$

$\left(m - \frac{6}{18}\right)^2 = -\frac{4}{9}$

(9) contd.

$$M - \frac{6}{18} = \pm \frac{2}{3}i$$

$$M = \frac{1}{3} \pm \frac{2}{3}i \quad \therefore GS \quad y = e^{\frac{1}{3}x}\left(A\cos\tfrac{2}{3}x + B\sin\tfrac{2}{3}x\right)$$

(10) $$\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$$

AQE $$M^2 + \sqrt{3}M + 3 = 0$$

$$\left(M + \frac{\sqrt{3}}{2}\right)^2 + 3 - \frac{3}{4} = 0$$

$$\left(M + \frac{\sqrt{3}}{2}\right)^2 = -\frac{9}{4}$$

$$M = -\frac{\sqrt{3}}{2} \pm \frac{3}{2}i \quad \therefore GS \quad y = e^{-\frac{\sqrt{3}}{2}}\left(A\cos\tfrac{3}{2}x + B\sin\tfrac{3}{2}x\right)$$

The Particular Integral

The second order linear differential equations that we have so far considered have been of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

We next need to consider second order equations in which the RHS is not zero, but a function of x, ie

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

Consider for example the equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^x \quad — (1)$$

RHS suggests solution of the form $y=\lambda e^x$ (because, if you diff once, then twice and subst in LHS, it will all contain e^x)

$\therefore \frac{dy}{dx}=\lambda e^x \quad \frac{d^2y}{dx^2}=\lambda e^x$

Sub in (1) $\quad \lambda e^x-5\lambda e^x+6\lambda e^x=e^x$

$2\lambda=1$

$\lambda=\frac{1}{2}$

$\therefore y=\frac{1}{2}e^x$

So we see that $y=\dfrac{1}{2}e^x$ is a solution of the given equation but it cannot be a complete solution because it contains no arbitrary constants. However it must be part of the complete solution, and is called a *particular integral* (P.I.).

The remainder of the solution can be found by considering the simpler differential equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

AQE $\quad m^2-5m+6=0$

$(m-2)(m-3)=0$

$m=2,3$

$\therefore$ General Solution $y=Ae^{2x}+Be^{3x}$

Clearly this solution alone does not satisfy the original equation, but, when combining it with the particular integral, we can show that:

If $y=Ae^{2x}+Be^{3x}+\frac{1}{2}e^x$

$\frac{dy}{dx}=2Ae^{2x}+3Be^{3x}+\frac{1}{2}e^x, \quad \frac{d^2y}{dx^2}=4Ae^{2x}+9Be^{3x}+\frac{1}{2}e^x$

And by eliminating the constants A and B that $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}+6y=e^x$ can't be bothered!

So the general solution of the given differential equation is

$$y = Ae^{2x} + Be^{3x} + \frac{1}{2}e^{x}$$

which is obtained by adding the complementary function $(Ae^{2x} + Be^{3x})$ and the particular integral.

In fact, for all differential equations of the type

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

the general solution is

$$y = CF + PI$$

Selecting the trial solution for the particular integral:
The following particular integrals should be learned. If the PI is non-standard (ie not from this list), then the PI would be given to you in the question.

Form of f(x)	Form of Particular Integral
k	λ
kx	$\lambda + \mu x$
kx^2	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$ or $n\sin\omega x$ or $m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

eg4 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \cos 2x$$

Exercise 5D Pg 97 Odds

The Failure Case

Consider the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^x$ — (1)

Complimentary function is: $m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$y = Ae^{4x} + Be^{x}$

Particular Integral of form: $y = \lambda e^{x}$

But λe^{x} is already included in the term Be^{x} in the C.F.
So $y = \lambda e^{x}$ satisfies the diff eq^n when RHS = 0
∴ cannot satisfy the diff eq^n when RHS $= e^{x}$

where the trial form of particular integral is a solution of the differential equation with RHS = 0, ie is part of the complementary function then we need to consider a different trial solution, which is found by multiplying the P.I. by x, or x^2...if necessary.

So in our example:

P.I. $y = \lambda x e^{x}$

$\frac{dy}{dx} = \lambda x e^{x} + \lambda e^{x}$, $\frac{d^2y}{dx^2} = (\lambda x e^{x} + \lambda e^{x}) + \lambda e^{x} = \lambda x e^{x} + 2\lambda e^{x}$

In ① $(\lambda x e^{x} + 2\lambda e^{x}) - 5(\lambda x e^{x} + \lambda e^{x}) + 4\lambda x e^{x} = e^{x}$

$-3\lambda = 1$

$\lambda = -\frac{1}{3}$ $\quad \therefore$ P.I.: $y = -\frac{1}{3}xe^{x}$

and General Solution $y = Ae^{4x} + Be^{x} - \frac{1}{3}xe^{x}$

Calculation of Arbitrary Constants

Because the solution of a second order differential equation contains two arbitrary constants, their evaluation requires two initial conditions.

eg5 Find y in terms of x given that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 12$$

and that $\frac{dy}{dx} = 1$ and $y = 0$ at $x = 0$

Exercise 5E Pg 99 Evens

Solving Second Order Differential Equations using a Change of Variable:

eg6 Use the substitution $x = e^{u}$ to find the general solution to the differential equation

$$x^2\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

Exercise 5F Pg 101 Evens

eg 4

$$\frac{d^2y}{dx^2} + y = \cos 2x \qquad \text{— (1)}$$

For complimentary function $\quad m^2 + 1 = 0$

$$m = \pm i$$

$\therefore$ C.F. $\quad \cancel{y = e^{x}(A+Bx)} \quad y = A\cos x + B\sin x$

Now particular integral $\quad y = \lambda \cos 2x + \mu \sin 2x$

$$\frac{dy}{dx} = -2\lambda \sin 2x + 2\mu \cos 2x$$

$$\frac{d^2y}{dx^2} = -4\lambda \cos 2x - 4\mu \sin 2x$$

Subst in (1) $\quad (-4\lambda \cos 2x - 4\mu \sin 2x) + (\lambda \cos 2x + \mu \sin 2x) = \cos 2x$

$$\cos 2x(-4\lambda + \lambda) + \sin 2x(-4\mu + \mu) = \cos 2x$$

Compare coeffs $\quad -4\lambda + \lambda = 1$

$$\lambda = -\tfrac{1}{3}$$

$$-4\mu + \mu = 0$$

$$\mu = 0$$

$\therefore$ PI $\quad y = -\frac{1}{3}\cos 2x$

Hence general solution $\quad \cancel{y = e^{x}(A+Bx) - \tfrac{1}{3}\sin 2x}$

$$y = A\cos x + B\sin x - \frac{1}{3}\cos 2x.$$

eg5 $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 12$ — ①

CF $m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$m = 1, 3$

$\therefore$ C.F. $y = Ae^x + Be^{3x}$

PI : $y = \lambda$ $\dfrac{dy}{dx} = 0$ $\dfrac{d^2y}{dx^2} = 0$

in ① $0 - 4(0) + 3\lambda = 12$

$\lambda = 4$

$\therefore$ General Solution $y = Ae^x + Be^{3x} + 4$ — ②

Now when $y = 0$, $x = 0$ $0 = A + B + 4$ $\Rightarrow$ ③ $A + B = -4$ — ③

when $x = 0$ $\dfrac{dy}{dx} = 1$ $\dfrac{dy}{dx} = Ae^x + 3Be^{3x}$

$1 = A + 3B$ — ④

④ − ③ $2B = 5$

$B = \frac{5}{2}$

in ③ $A + \frac{5}{2} = -4$

$A = -4 - \frac{5}{2} = -\frac{13}{2}$

$\therefore$ $y = \frac{5}{2}e^{3x} - \frac{13}{2}e^x + 4$.

Ex 5D

(1) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10$ — (1)

For complementary function, $m^2 + 6m + 5 = 0$

$(m+5)(m+1) = 0$

$m = -1, -5$

$\therefore$ C.F. $y = Ae^{-x} + Be^{-5x}$

Now Particular integral $y = k$ $\quad \frac{dy}{dx} = 0 \quad \frac{d^2y}{dx^2} = 0$

in (1) $\quad 0 + 6(0) + 5k = 10$

$k = 2$

$\therefore$ General Solution $\quad y = Ae^{-x} + Be^{-5x} + 2$

(2) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$ — (1)

CF $\quad m^2 - 8m + 12 = 0$

$(m-6)(m-2) = 0$

$m = 2, 6$

$\therefore$ CF $\quad y = Ae^{2x} + Be^{6x}$

Now PI $\quad y = ax + b \quad \frac{dy}{dx} = a \quad \frac{d^2y}{dx^2} = 0$

in (1) $\quad 0 - 8a + 12(ax+b) = 36x$

$-8a + 12ax + 12b = 36x$

Compare coefficients $\quad 12a = 36 \quad\quad -8a + 12b = 0$

$a = 3 \quad\quad 12b = 24$

$b = 2$

$\therefore$ General Solution $y = Ae^{2x} + Be^{6x} + 3x + 2$

(3) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 12e^{2x}$ — (1)

C.F. $M^2 + M - 12 = 0$

$(M+4)(M-3) = 0$

$M = -4, 3$

$\therefore$ CF $y = Ae^{-4x} + Be^{3x}$

Now PI $y = ke^{2x}$ $\frac{dy}{dx} = 2ke^{2x}$ $\frac{d^2y}{dx^2} = 4ke^{2x}$

in (1) $4k\cancel{e^{2x}} + 2k\cancel{e^{2x}} - 12k\cancel{e^{2x}} = 12\cancel{e^{2x}}$

$-6k = 12$

$k = -2$

$\therefore$ General Solution $y = Ae^{-4x} + Be^{3x} - 2e^{2x}$

(4) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 5$ — (1)

C.F. $M^2 + 2M - 15 = 0$

$(M+5)(M-3) = 0$

$M = -5, 3$

$\therefore$ CF $y = Ae^{-5x} + Be^{3x}$

Now P.I $y = k$ $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$

in (1) $-15k = 5$

$k = -\frac{1}{3}$

$\therefore$ General Solution $y = Ae^{-5x} + Be^{3x} - \frac{1}{3}$

(5) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12$ — (1)

CF $m^2 - 8m + 16 = 0$
$(m-4)(m-4) = 0$
$m = 4$

$\therefore$ CF $y = e^{4x}(A + Bx)$

Now P.I. $y = \lambda + \mu x$ $\quad \frac{dy}{dx} = \mu \quad \frac{d^2y}{dx^2} = 0$

in (1) $0 - 8\mu + 16(\lambda + \mu x) = 8x + 12$
$(16\lambda - 8\mu) + 16\mu x = 8x + 12$

Compare coef's $16\mu = 8$
$\mu = \frac{1}{2}$

$16\lambda - 8 \times \frac{1}{2} = 12$
$16\lambda = 16$
$\lambda = 1$

$\therefore$ General Solution $y = e^{4x}(A + Bx) + \frac{1}{2}x + 1$

(6) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$ — (1)

CF $m^2 + 2m + 1 = 0$
$(m+1)(m+1) = 0$
$m = -1$

$\therefore$ CF $y = e^{-x}(A + Bx)$

Now P.I $y = \lambda \cos 2x + \mu \sin 2x$
$\frac{dy}{dx} = -2\lambda \sin 2x + 2\mu \cos 2x$
$\frac{d^2y}{dx^2} = -4\lambda \cos 2x - 4\mu \sin 2x$

(6) Contd in (1) $-4\lambda \cos 2x - 4\mu \sin 2x + 2(-2\lambda \sin 2x + 2\mu \cos 2x)$
$+ \lambda \cos 2x + \mu \sin 2x = 25 \cos 2x$

$\cos 2x(-4\lambda + 4\mu + \lambda) + \sin 2x(-4\mu - 4\lambda + \mu) = 25 \cos 2x$

Compare coefficients.

$4\mu - 3\lambda = 25$ — (1)
$-3\mu - 4\lambda = 0$ — (2)

From (2) $\mu = -\frac{4\lambda}{3}$ — (3)

in (1) $4\left(-\frac{4\lambda}{3}\right) - 3\lambda = 25$
$-\frac{16}{3}\lambda - 3\lambda = 25$
$-\frac{25}{3}\lambda = 25$
$\lambda = -3$

in (3) $\mu = 4$

$\therefore$ General Solution $y = e^{-x}(A + Bx) - 3\cos 2x + 4\sin 2x$

(7) $\frac{d^2y}{dx^2} + 81y = 15e^{3x}$ — (1)

CF $m^2 + 81 = 0$
$m = \pm 9i$

$\therefore$ CF $y = A\sin 9x + B\sin 9x$

Now PI: $y = ke^{3x}$ $\frac{dy}{dx} = 3ke^{3x}$ $\frac{d^2y}{dx^2} = 9ke^{3x}$

in (1) $9ke^{3x} + 81ke^{3x} = 15e^{2x}$
$90k = 15$
$k = \frac{15}{90} = \frac{1}{6}$

(7) contd $\therefore$ General Solution $y = A\sin 9x + B\cos 9x + \frac{1}{6}e^{3x}$

(8) $\frac{d^2y}{dx^2} + 4y = \sin x$ — (1)

CF $m^2 + 4 = 0$

$m = \pm 2i$

$\therefore$ CF is $y = A\sin 2x + B\cos 2x$

Now PI: $y = \lambda\cos x + \mu\sin x$

$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$

$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$

in (1) $-\lambda\cos x - \mu\sin x + 4(\lambda\cos x + \mu\sin x) = \sin x$

$\cos x(-\lambda + 4\lambda) + \sin x(-\mu + 4\mu) = \sin x$

Compare coeff's $3\lambda = 0$ $\lambda = 0$

$3\mu = 1$

$\mu = \frac{1}{3}$

$\therefore$ General Solution $y = A\sin 2x + B\cos 2x + \frac{1}{3}\sin x$

(9) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$ — (1)

CF $m^2 - 4m + 5 = 0$

$(m-2)^2 + 1 = 0$

$m - 2 = \pm i$

$m = 2 \pm i$

$\therefore$ CF $y = e^{2x}(A\cos x + B\sin x)$

Now PI: $y = \lambda + \mu x + \nu x^2$

$\frac{dy}{dx} = \mu + 2\nu x$

$\frac{d^2y}{dx^2} = 2\nu$

in (1) $2\nu - 4(\mu + 2\nu x) + 5(\lambda + \mu x + \nu x^2) = 25x^2 - 7$

$5\nu x^2 + x(-8\nu + 5\mu) + (2\nu - 4\mu + 5\lambda) = 25x^2 - 7$

Comp Coeffs

$5\nu = 25$

$\nu = 5$

$-8\nu + 5\mu = 0$

$-40 + 5\mu = 0$

$\mu = 8$

$2\nu - 4\mu + 5\lambda = -7$

$10 - 32 + 5\lambda = -7$

$5\lambda = 15$

$\lambda = 3$

$\therefore$ General Solution $y = e^{2x}(A\cos x + B\sin x) + 3 + 8x + 5x^2$

(10) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 26y = e^x$ — (1)

CF: $m^2 - 2m + 26 = 0$

$(m-1)^2 + 25 = 0$

$m = 1 \pm 5i$

$\therefore \; y = e^x(A\cos 5x + B\sin 5x)$

Ae^x not part of CF
$(A\cos 5x)e^x$ is etc.

Now particular integral cannot be Ae^x as this is part of CF.

So let $y = kxe^x$

$\dfrac{dy}{dx} = kxe^x + ke^x$

$\dfrac{d^2y}{dx^2} = (kxe^x + ke^x) + ke^x = kxe^x + 2ke^x$

in (1)

$kxe^x + 2ke^x - 2(kxe^x + ke^x) + 26kxe^x = e^x$

$\div e^x$

$kx + 2k - 2kx - 2k + 26kx = 1$

$25kx = 1.$

Complete

PI $\quad y = Ae^x \quad \dfrac{dy}{dx} = Ae^x \quad \dfrac{d^2y}{dx^2} = Ae^x$

in (1) $\quad Ae^x - 2Ae^x + 26Ae^x = e^x$

$25A = 1$

$A = \dfrac{1}{25}$

$\therefore$ General Solution $\quad y = e^x(A\cos 5x + B\sin 5x) + \dfrac{1}{25}e^x$

11 (a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ — (1)

Now For P.I. $y = Ax^2e^x$

$$\frac{dy}{dx} = Ax^2.e^x + 2Axe^x$$

$$\frac{d^2y}{dx^2} = (Ax^2.e^x + e^x.2Ax) + (2Axe^x + 2Ae^x)$$

In (1) $(Ax^2e^x + 4Axe^x + 2Ae^x) - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = e^x$

$\div e^x$

$$x^2(A - 2\cancel{A^2}^{+A}) + x(4A - 4A) + (2A) = 1$$

$$2A = 1$$
$$A = 1/2$$

(b) CF. $M^2 - 2M + 1 = 0$
$(M-1)(M-1) = 0$

$\therefore$ CF $y = e^x(A + Bx)$

Hence General Solution $y = e^x(A+Bx) + \frac{1}{2}x^2e^x$

Ex 5E

① $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^{x}$ — ①

CF: $M^2 + 5M + 6 = 0$

$(M+2)(M+3) = 0$

$M = -2, -3$

$\therefore$ CF: $y = Ae^{-2x} + Be^{-3x}$

PI: Let $y = \lambda e^{x}$

$\frac{dy}{dx} = \frac{d^2y}{dx^2} = \lambda e^{x}$

in ① $\lambda e^{x} + 5\lambda e^{x} + 6\lambda e^{x} = 12e^{x}$

$\lambda = 1$.

$\therefore$ General Solution $y = Ae^{-2x} + Be^{-3x} + e^{x}$

@ $x=0, y=1$ $\quad 1 = A + B + 1 \Rightarrow A + B = 0$ — ②

@ $x=0, \frac{dy}{dx} = 0$ $\quad \frac{dy}{dx} = -2Ae^{-2x} - 3Be^{-3x} + e^{x}$

$0 = -2A - 3B + 1 \Rightarrow 2A + 3B = 1$ — ③

from ② $A = -B$

in ③ $-2B + 3B = 1$

$B = 1 \quad \therefore A = -1$

$\therefore$ Solution $y = e^{-3x} - e^{-2x} + e^{x}$

② $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x}$ — ①

CF: $m^2 + 2m = 0$

$m(m+2) = 0$

$m=0 \quad m=-2$

$\therefore$ CF $y = A + Be^{-2x}$

PI Let $y = \lambda e^{2x}$ $\quad \frac{dy}{dx} = 2\lambda e^{2x}$, $\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

in ① $4\lambda e^{2x} + 4\lambda e^{2x} = 12e^{2x}$

$8\lambda = 12$

$\lambda = \frac{3}{2}$

$\therefore$ General Solution $y = A + Be^{-2x} + \frac{3}{2}e^{2x}$ — ②

@ $x=0, y=2$ in ② $\quad 2 = A + B + \frac{3}{2} \Rightarrow A + B = \frac{1}{2}$ — ①

@ $x=0, \frac{dy}{dx} = 6$, $\quad \frac{dy}{dx} = -2Be^{-2x} + 3e^{2x}$

$\therefore 6 = -2B + 3 \Rightarrow B = -\frac{3}{2}$

in ① $A - \frac{3}{2} = \frac{1}{2}$

$\therefore A = 2$

$\therefore$ Particular Solution $y = 2 - \frac{3}{2}e^{-2x} + \frac{3}{2}e^{2x}$

(3) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14$ — (1)

CF: $M^2 - M - 42 = 0$

$(M-7)(M+6) = 0$

$M = -6, 7$

$\therefore$ CF $y = Ae^{-6x} + Be^{7x}$

PI: Let $y = \lambda$ $\quad \frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$

In (1) $-42\lambda = 14$

$A = \frac{-14}{42} = -\frac{1}{3}$

$\therefore$ General Solution $y = Ae^{-6x} + Be^{7x} - \frac{1}{3}$ — (2)

Now @ $x=0$, $y=0$ in (2) $\quad 0 = A + B - \frac{1}{3} \implies A + B = \frac{1}{3}$ — (3)

@ $x=0$, $\frac{dy}{dx} = \frac{1}{6}$ $\quad$ From (2) $\frac{dy}{dx} = -6Ae^{-6x} + 7Be^{7x}$

$\frac{1}{6} = -6A + 7B \implies 42B - 36A = 1$ — (4)

From (3) $B = \frac{1}{3} - A$

in (4) $42\left(\frac{1}{3} - A\right) - 36A = 1$

$14 - 42A - 36A = 1$

$13 = 78A$

$A = \frac{13}{78} = \frac{1}{6} \quad \therefore \quad B = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

Hence particular solution $y = \frac{1}{6}e^{-6x} + \frac{1}{6}e^{7x} - \frac{1}{3}$

④ $\frac{d^2y}{dx^2} + 9y = 16\sin x$ — ①

CF: $m^2 + 9 = 0$

$m = \pm 3i$

$\therefore$ CF $y = A\cos 3x + B\sin 3x$

PI: let $y = \lambda\cos x + \mu\sin x$

$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$

$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$

in ① $(-\lambda\cos x - \mu\sin x) + 9(\lambda\cos x + \mu\sin x) = 16\sin x$

$\cos x(-\lambda + 9\lambda) + \sin x(-\mu + 9\mu) = 16\sin x$

Compare coeff's $\cos x$: $8\lambda = 0$ $\quad$ $\sin x$: $8\mu = 16$

$\lambda = 0$ $\quad$ $\mu = 2$

$\therefore$ General Solution $y = A\cos 3x + B\sin 3x + 2\sin x$

@ $x=0$ $y=1$ $\quad 1 = A + 0 + 0 \quad \therefore A = 1$

@ $x=0$ $\frac{dy}{dx} = 8$ $\quad \frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + 2\cos x$

$8 = 0 + 3B + 2$

$B = 2$

$\therefore$ particular solution $y = \cos 3x + 2\sin 3x + 2\sin x$

⑤ $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x$

CF: $4m^2 + 4m + 5 = 0$

$\div 4$ $\quad m^2 + m + \frac{5}{4} = 0$

$(m + \frac{1}{2})^2 + 1 = 0$

$m = -\frac{1}{2} \pm i$

$\therefore$ C.F: $\quad y = e^{-\frac{1}{2}x}(A\cos x + B\sin x)$

PI: let $y =$

(5) $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x$ — (1)

$\therefore$ CF: $4m^2 + 4m + 5 = 0$

$\div 4$

$m^2 + m + \frac{5}{4} = 0$

$\left(m+\frac{1}{2}\right)^2 + 1 = 0$

$m = -\frac{1}{2} \pm i$

$\therefore$ CF $y = e^{-\frac{1}{2}x}(A\cos x + B\sin x)$

PI: Let $y = \lambda\cos x + \mu\sin x$

$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$

$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$

in (1) $4(-\lambda\cos x - \mu\sin x) + 4(-\lambda\sin x + \mu\cos x) + 5(\lambda\cos x + \mu\sin x) = \sin x + 4\cos x$

$\cos x(-4\lambda + 4\mu + 5\lambda) + \sin x(-4\mu - 4\lambda + 5\mu) = \sin x + 4\cos x$

Compare coeff's $\lambda + 4\mu = 4$ — (1)

$-4\lambda + \mu = 1$ — (2)

From (1) $\lambda = 4 - 4\mu$

in (2) $-4(4-4\mu) + \mu = 1$

$-16 + 16\mu + \mu = 1$

$\mu = 1$

$\therefore \lambda = 0$

$\therefore$ General Solution $y = e^{-\frac{1}{2}x}(A\cos x + B\sin x) + \sin x$

(5) <u>contd</u> when $x=0, y=0$ $\quad 0 = A$

$$\therefore \quad y = Be^{-\frac{1}{2}x}\sin x + \sin x$$

$$\frac{dy}{dx} = Be^{-\frac{1}{2}x}.\cos x + -\frac{1}{2}Be^{-\frac{1}{2}x}\sin x + \cos x$$

when $x=0$ $\frac{dy}{dx}=0$ $\quad 0 = B - 0 + 1$

$B = -1$

$$\therefore \quad y = \sin x\left(1 - e^{-\frac{1}{2}x}\right)$$

(6) $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3$ $\quad$ — (1)

CF: $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$\therefore$ CF $\cancel{x}$ $y = Ae^{t} + Be^{2t}$

Now P.I. try $x = \lambda + \mu t$

$\frac{dx}{dt} = \mu \quad \frac{d^2x}{dt^2} = 0$

in (1) $\quad 0 - 3\mu + 2(\lambda + \mu t) = 2t - 3$

$2\mu t + (2\lambda - 3\mu) = 2t - 3$

Comp coeff's $\quad 2\mu = 2 \qquad 2\lambda - 3\mu = -3$

$\mu = 1 \qquad 2\lambda - 3 = -3$

$\lambda = 0$

$\therefore$ General Solution $\quad x = Ae^{t} + Be^{2t} + t$ $\quad$ — (2)

(6) $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3 \quad — ①$

CF $\quad m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$\therefore$ CF $\quad \cancel{y = Ae^x + Be^{2x}} \quad x = Ae^t + Be^{2t}$

PI $\quad$ Let $\cancel{y =}\ x = \lambda + \mu t \qquad \frac{dx}{dt} = \mu \qquad \frac{d^2x}{dt^2} = 0$

in ① $\quad 0 - 3\mu + 2(\lambda + \mu t) = 2t - 3$

comp coeff's $\quad 2\mu = 2 \qquad 2\lambda - 3\mu = -3$

$\mu = 1 \qquad 2\lambda - 3 = -3$

$\lambda = 0$

$\therefore$ General Solution $\quad x = Ae^t + Be^{2t} + t$

Now when $\cancel{x}\ t = 0, x = 2 \qquad 2 = A + B \quad — ①$

when $t = 0 \quad \frac{dx}{dt} = 4 \qquad \frac{dx}{dt} = Ae^t + 2Be^{2t} + 1$

$4 = A + 2B + 1 \qquad A + 2B = 3 \quad — ②$

② − ① $\quad B = 1$

in ① $\quad A = 1$

$\therefore$ particular solution $\quad x = e^t + e^{2t} + t \qquad$ * diff to book but is correct.

(7) $\frac{d^2x}{dt^2} - 9x = 10\sin t$ — (1)

CE $m^2 - 9 = 0$

$m = \pm 3$

$\therefore$ CF: $x = Ae^{3t} + Be^{-3t}$

PI try $x = \lambda \cos t + \mu \sin t$

$\frac{dx}{dt} = -\lambda \sin t + \mu \cos t$

$\frac{d^2x}{dt^2} = -\lambda \cos t - \mu \sin t$

in (1) $-\lambda \cos t - \mu \sin t - 9(\lambda \cos t + \mu \sin t) = 10\sin t$

$-10\lambda \cos t - 10\mu \sin t = 10\sin t$

Comp coeff's $\lambda = 0 \quad \mu = -1$

$\therefore$ General Solution $x = Ae^{3t} + Be^{-3t} - \sin t$

Now when $t=0$, $x=2$ $\quad 2 = A + B$ — (1)

$\frac{dx}{dt} = 3Ae^{3t} - 3Be^{-3t} - \cos t$

when $t=0$ $\frac{dx}{dt} = -1$ $\quad -1 = 3A - 3B - 1$

$A - B = 0$ — (2)

(1) + (2) $2A = 2 \quad A = 1, B = 1$

$\therefore$ particular solution $x = e^{3t} + e^{-3t} - \sin t$

(8) $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t}$ — (1)

CE $M^2 - 4M + 4 = 0$

$(M-2)(M-2) = 0$

$\therefore$ CE $y = e^{2t}(A + Bt)$

PI try $x = \lambda t^3 e^{2t}$

$\frac{dx}{dt} = 2\lambda t^3 e^{2t} + 3\lambda t^2 e^{2t}$

$\frac{d^2x}{dt^2} = 4\lambda t^3 e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t e^{2t}$

in (1) $(4\lambda t^3 + 6\lambda t^2 + 6\lambda t^2 + 6\lambda t) - 4(2\lambda t^3 + 3\lambda t^2) + 4\lambda t^3 = 3t$

$t^3(4\lambda - 8\lambda + 4\lambda) + t^2(12\lambda - 12\lambda) + 6\lambda t = 3t$

$\lambda = \frac{1}{2}$

$\therefore$ General Solution $x = e^{2t}(A + Bt) + \frac{1}{2}t^3 e^{2t}$

Now when $x = 0$ $t = 0$ $0 = A$

$\therefore$ $x = Bte^{2t} + \frac{1}{2}t^3 e^{2t}$

$\frac{dx}{dt} = 2Bte^{2t} + Be^{2t} + t^3 e^{2t} + \frac{3}{2}t^2 e^{2t}$

when $t = 0$ $\frac{dx}{dt} = 1$ $1 = 0 + B + 0 + 0$

$B = 1$

$\therefore$ $x = e^{2t}(t + \frac{1}{2}t^3)$

(9) $25\frac{d^2x}{dt^2} + 36x = 18 \quad -(1)$

CF $\quad 25m^2 + 36 = 0$

$M^2 = -\frac{36}{25}$

$M = \pm\frac{6i}{5}$

$\therefore$ CF $\cancel{y} = A\cos\frac{6}{5}t + B\sin\frac{6t}{5}$

PI try $x = \lambda \quad \frac{dx}{dt} = \frac{d^2x}{dt^2} = 0$

in (1) $\quad 36\lambda = 18$

$\lambda = \frac{1}{2}$

$\therefore$ General Solution $\quad x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t + \frac{1}{2}$

Now when $t=0, x=1 \quad 1 = A + \frac{1}{2} \Rightarrow A = \frac{1}{2}$

$x = \frac{1}{2}\cos\frac{6}{5}t + B\sin\frac{6}{5}t + \frac{1}{2}$

$\frac{dx}{dt} = -\frac{6}{10}\sin\frac{6}{5}t + \frac{6}{5}B\cos\frac{6}{5}t$

when $t=0 \quad \frac{dx}{dt} = 0.6$

$0.6 = \frac{6}{5}B \quad B = \frac{1}{2}$

$\therefore$ particular solution $\cancel{y = \frac{1}{2}}$

$x = \frac{1}{2}\left(\cos\frac{6}{5}t + \sin\frac{6t}{5} + 1\right)$

(10) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2$ — (1)

CF $m^2 - 2m + 2 = 0$

$(m-1)^2 + 1 = 0$

$(m-1)^2 = -1$

$m = 1 \pm i$

$\therefore$ CF: $x = e^t(A\cos t + B\sin t)$

PI try $x = \lambda + \mu t + \nu t^2$

$\frac{dx}{dt} = \mu + 2\nu t$

$\frac{d^2x}{dt^2} = 2\nu$

in (1) $2\nu - 2(\mu + 2\nu t) + 2(\lambda + \mu t + \nu t^2) = 2t^2$

$t^2(2\nu) + t(-4\nu + 2\mu) + (2\nu - 2\mu + 2\lambda) = 2t^2$

Comp coef's $2\nu = 2$

$\nu = 1$

$-4\nu + 2\mu = 0$

$\mu = 2$

$2\nu - 2\mu + 2\lambda = 0$

$2 - 4 + 2\lambda = 0$

$\lambda = 1$

$\therefore$ General Solution $x = e^t(A\cos t + B\sin t) + 1 + 2t + t^2$

(10) cont'd when $t=0, x=1$ $\quad 1 = A + 1$

$A = 0$

$\therefore \quad x = Be^t \sin t + 1 + 2t + t^2$

$\frac{dx}{dt} = Be^t \cos t + Be^t \sin t + 2 + 2t$

when $t=0$ $\frac{dx}{dt} = 3$ $\quad 3 = B + 2$

$B = 1$

$\therefore$ particular solution $x = e^t \sin t + 1 + 2t + t^2$

Solving Second Order Differential Equations using a Change of Variable:

These can be subdivided into two different types:

- Using the substitution to replace the independent variable.

<u>eg6</u> Use the substitution $x = e^u$ to find the general solution to the differential equation

$$x^2\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

<u>Exercise 5F</u> Pg 101 Odds (NOT Q7)

- Using the substitution to replace the dependent variable.

<u>eg7</u> (a) Show that the substitution $v = xy$ transforms the differential equation

$$x\frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} + 4(1+x)y = 32e^{2x}, x \neq 0$$

into the differential equation

$$\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v = 32e^{2x}$$

(b) Given that $v = ae^{2x}$, where a is constant, is a particular integral of this transformed equation, find a.

(c) Find the solution of the differential equation

$$x\frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} + 4(1+x)y = 32e^{2x}, x \neq 0$$

for which $y = 2e^2$ and $\frac{dy}{dx} = 0$ at $x = 1$

(d) Determine whether or not this solution remains finite as $x \to \infty$.

<u>Exercise 5F</u> Pg 101 Q's 7 & 8

eg. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ — (1)

Need constant coef's so can't solve.

Using subst $x = e^u$, we need terms for $\frac{dy}{du}$ and $\frac{d^2y}{du^2}$:

(a) $\frac{dy}{du}$: Using chain rule $\frac{dy}{du} = \frac{dy}{\bigcirc} \cdot \frac{\bigcirc}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$

Now $\frac{dx}{du} = e^u$ $\therefore$ $\frac{dy}{du} = e^u \frac{dy}{dx}$

but $x = e^u$ $\therefore$ $\frac{dy}{du} = x\frac{dy}{dx}$ $\Rightarrow$ middle term of diff eqⁿ.

(b) Now $\frac{d^2y}{du^2} = \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(\underset{p}{e^u}\ \underset{q}{\frac{dy}{dx}}\right)$

Using product rule $\frac{d^2y}{du^2} = p\frac{dq}{du} + q\frac{dp}{du}$

$= e^u\left(\frac{dq}{dx}\cdot\frac{dx}{du}\right) + \left(\frac{dy}{dx}\right)e^u$

$= e^u\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)e^u + e^u\frac{dy}{dx}$

Learn this

NASTY!

but $e^u = x$ $\therefore$ $\frac{d^2y}{du^2} = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$

also $x\frac{dy}{dx} = \frac{dy}{du}$ $\therefore$ $\frac{d^2y}{du^2} = x^2\frac{d^2y}{dx^2} + \frac{dy}{du}$

$\therefore$ $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ $\Rightarrow$ 1st term of diff eqⁿ.

Q7b Contd sub in ①

$$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + \frac{dy}{du} + y = 0$$

$$\frac{d^2y}{du^2} + y = 0$$

AQE $m^2 + 1 = 0$

$m = \pm i$

$\therefore$ General Solution $y = A\cos u + B\sin u$

but if $x = e^u$ $u = \ln|x|$

$\therefore y = A\cos(\ln|x|) + B\sin(\ln|x|)$.

Q7 (a) $x\dfrac{d^2y}{dx^2} + 2(1+2x)\dfrac{dy}{dx} + 4(1+x)y = 32e^{2x}$ — ①

$v = xy$

$\dfrac{dv}{dx} = \dfrac{d}{dx}(xy) = x \cdot \dfrac{dy}{dx} + y$

$\dfrac{d^2v}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dv}{dx}\right) = \dfrac{d}{dx}\left(x\dfrac{dy}{dx} + y\right)$

$= x \cdot \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) + \dfrac{dy}{dx} \cdot \dfrac{d}{dx}(x) + \dfrac{d}{dx}(y)$

$= x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{dy}{dx}$

$\dfrac{d^2v}{dx^2} = x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}$

From ① $\left(x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}\right) + 4\left(x\dfrac{dy}{dx} + y\right) + 4xy = 32e^{2x}$

$\therefore \quad \dfrac{d^2v}{dx^2} + 4\dfrac{dv}{dx} + 4v = 32e^{2x}$ — ② As required.

(b) If particular integral $v = ae^{2x}$ $\quad \dfrac{dv}{dx} = 2ae^{2x}$ $\quad \dfrac{d^2v}{dx^2} = 4ae^{2x}$

in ② $\quad 4ae^{2x} + 4(2ae^{2x}) + 4ae^{2x} = 32e^{2x}$

$16a = 32$

$a = 2$

$\therefore$ particular integral $v = 2e^{2x}$

Q7 (c) C.F. $m^2+4m+4=0$

$(m+2)(m+2)=0$

$m=-2$

repeated roots $\therefore$ $v = e^{-2x}(A+Bx)$

$\therefore$ General Solution $v = e^{-2x}(A+Bx) + 2e^{2x}$

but $v = xy$ $\therefore$ $y = \dfrac{e^{-2x}(A+Bx)+2e^{2x}}{x}$

Now when $x=1$, $y=2e^2$

$$2e^2 = e^{-2}(A+B) + 2e^2$$

$$A = -B.$$

$$\therefore y = \frac{2e^{2x} - Be^{-2x}(x-1)}{x} \quad (3)$$

$$\frac{dy}{dx} = \frac{x\left[4e^{2x} - \left(Be^{-2x}.1 + -2Be^{-2x}(x-1)\right)\right] - \left[2e^{2x} - Be^{-2x}(x-1)\right].1}{x^2}$$

when $x=1$, $\frac{dy}{dx}=0$

$$0 = 4e^2 - Be^{-2} + 0 - 2e^2$$

$$Be^{-2} = 2e^2$$

$$\frac{B}{e^2} = 2e^2$$

$$B = 2e^2.e^2 = 2e^4$$

in (3) $yx = 2e^{2x} - 2e^4e^{-2x}(x-1)$

$$y = \frac{2}{x}e^{2x} - 2e^4\left(\frac{x}{x} - \frac{1}{x}\right)e^{-2x}$$

$$y = 2e^4\left(\frac{1}{x} - 1\right) + \frac{2}{x}e^{2x}$$

(d) $y = 2e^4\left(\frac{1}{x} - 1\right) + \frac{2}{x}e^{2x}$

as $x \to \infty$ $\quad \frac{1}{x} - 1 \to -1 \quad \therefore \quad 2e^4\left(\frac{1}{x} - 1\right) \Rightarrow -2e^4$ ie finite

as $x \to \infty$ $\quad \frac{2}{x} \to 0 \quad e^{2x} \to \infty \quad \therefore \quad \frac{2}{x} \times e^{2x} \Rightarrow \infty$ ie infinite

~~e^{2x} moves it quicker towards ∞ than $\frac{2}{x}$~~

but $\frac{2}{x} \neq 0 \quad \therefore$ solution doesn't remain finite.

Summary Ex 5F

Q's 1→6 given: $x = e^u$

turn $\frac{dy}{dx} \Rightarrow \frac{dy}{du}$ change bot var

(7) given $y = \frac{z}{x}$ turn $\frac{dy}{dx} \Rightarrow \frac{dz}{dx}$ change top var

(8) given $y = \frac{z}{x^2}$ turn $\frac{dy}{dx} \rightarrow \frac{dz}{dx}$ change top var.

(9) given $z = \sin x$ turn $\frac{dy}{dx} \Rightarrow \frac{dy}{dz}$ change bot var.

- When changing top variable, rearrange subst into topvar = f(bottvar)$_x$ oth var then diff & double diff
- when changing bott variable, need to use chain rule to get $\frac{d}{d}$ form required.

Ex 5F

In Q's 1→6, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ & $x\frac{dy}{dx} = \frac{dy}{du}$

① $x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0$

$\frac{d^2y}{du^2} - \frac{dy}{du} + 6\left(\frac{dy}{du}\right) + 4y = 0$

$\frac{d^2y}{du^2} + 5\frac{dy}{du} + 4y = 0$

AQE $m^2 + 5m + 4 = 0$

$(m+1)(m+4) = 0$

$m = -1, -4$

$\therefore \quad y = Ae^{-u} + Be^{-4u}$

but $u = \ln x$

$\therefore \quad y = Ae^{\ln x^{-1}} + Be^{\ln x^{-4}}$

$y = \frac{A}{x} + \frac{B}{x^4}$

② $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$

$\frac{d^2y}{du^2} - \frac{dy}{du} + 5\frac{dy}{du} + 4y = 0$

$\frac{d^2y}{du^2} + 4\frac{dy}{du} + 4y = 0$

AQE $m^2 + 4m + 4 = 0$

$m = -2, -2$

$y = e^{-2u}(A + Bu)$

② contd $u = \ln x$

$$y = e^{\ln x^{-2}}(A + B\ln|x|)$$

$$y = \frac{1}{x^2}(A + B\ln|x|)$$

③ $$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + \frac{6dy}{du} + 6y = 0$$

$$\frac{d^2y}{du^2} + 5\frac{dy}{du} + 6y = 0$$

AQE: $(m+3)(m+2) = 0$

$m = -2, -3$

$$y = Ae^{-2u} + Be^{-3u}$$

$u = \ln|x|$

$$y = Ae^{\ln x^{-2}} + Be^{\ln x^{-3}}$$

$$y = \frac{A}{x^2} + \frac{B}{x^3}$$

④ $$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0$$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 4\frac{dy}{du} - 28y = 0$$

$$\frac{d^2y}{du^2} + 3\frac{dy}{du} - 28y = 0$$

AQE $m^2 + 3m - 28 = 0$

$(m+7)(m-4) = 0$

$m = 4, -7$

4 Cont'd $\quad y = Ae^{-7u} + Be^{4u}$

$u = \ln x$

$$y = Ae^{\ln x^{-7}} + Be^{\ln x^{4}}$$

$$y = \frac{A}{x^7} + Bx^4$$

5 $\quad x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0$

$$\frac{d^2y}{du^2} - \frac{dy}{du} - 4\frac{du}{du} - 14y = 0$$

$$\frac{d^2y}{du^2} - 5\frac{dy}{du} - 14y = 0$$

AQE $\quad m^2 - 5m - 14 = 0$

$$(m-7)(m+2) = 0$$

$$m = -2, 7$$

$$y = Ae^{-2u} + Be^{7u}$$

$u = \ln|x|$

$$y = Ae^{\ln x^{-2}} + Be^{\ln x^{+7}}$$

$$y = \frac{A}{x^2} + Bx^7$$

(6) $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0$

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 2y = 0$$

$$\frac{d^2y}{du^2} + 2\frac{dy}{du} + 2y = 0$$

AOE $\quad m^2 + 2m + 2 = 0$

$$(m+1)^2 + 1 = 0$$

$$m = -1 \pm i$$

$$\therefore \quad y = e^{-u}(A\cos u + B\sin u)$$

$u = \ln|x|$

$$y = e^{\ln x^{-1}}(A\cos(\ln|x|) + B\sin(\ln|x|))$$

$$y = \frac{1}{x}(A\cos(\ln|x|) + B\sin(\ln|x|))$$

(7) $x\dfrac{d^2y}{dx^2} + (2-4x)\dfrac{dy}{dx} - 4y = 0$ —— (1)

If $y = \dfrac{z}{x}$ I need $\dfrac{dz}{dx}$ and $\dfrac{d^2z}{dx^2}$

(a) $\cancel{\dfrac{dz}{dx} = \dfrac{dz}{d\,} \cdot \dfrac{d\,}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx}}$

$y = \dfrac{z}{x}$ $z = xy$ $\cancel{\dfrac{dz}{dy} = \dfrac{d}{dy}(xy) =}$

$\cancel{= x\dfrac{d}{dy}(y) + y\dfrac{d}{dy}(x)}$

$\cancel{\dfrac{dz}{dy} = x + y\dfrac{dx}{dy}}$

$$\frac{dz}{dx} = \frac{d}{dx}(xy)$$

$$= x \cdot \frac{d}{dx}(y) + y\frac{d}{dx}(x)$$

$$\frac{dz}{dx} = x\frac{dy}{dx} + y \quad \text{—— (2)}$$

(b) $\dfrac{d^2z}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dz}{dx}\right) = \dfrac{d}{dx}\left(x\dfrac{dy}{dx} + y\right) = \cancel{x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}}$

prod rule

$$= \left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) + \frac{dy}{dx}$$

$$\frac{d^2z}{dx^2} = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} \quad \text{—— (3)}$$

Now from (1) $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 4x\dfrac{dy}{dx} - 4y = 0$

Sub in (3) & (4) $\dfrac{d^2z}{dx^2} - 4\left(\dfrac{dz}{dx} - y\right) - 4y = 0$

(7) <u>contd</u> $\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0$

AQE $\quad m^2 - 4m = 0$

$m = 0, \ m = 4$

$\cancel{y = A + Be} \quad z = A + Be^{4x}$

but $\cancel{z = 4}$ $z = yx$ $\therefore$ $y = \frac{A + Be^{4x}}{x}$

(8) $x^2\frac{d^2y}{dx^2} + 2x(x+2)\frac{dy}{dx} + 2(x+1)^2y = e^{-x}$ — (1)

if $y = \frac{z}{x^2}$, I need $\frac{dz}{dx}$ & $\frac{d^2z}{dx^2}$

(a) $\cancel{\frac{dz}{dx} = \frac{dz}{d} \cdot \frac{d}{dx} \quad \frac{dz}{dy} \frac{dy}{dx}}$

$z = x^2y \qquad \frac{dz}{dx} = x^2 \cdot \frac{dy}{dx} + 2xy$

(b) $\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(x^2\frac{dy}{dx} + 2xy\right)$

prod rules.

$= \left(x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}\right) + \left(2x \cdot \frac{dy}{dx} + 2y\right)$

$\frac{d^2z}{dx^2} = x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y$

From (1) $\quad x^2\frac{d^2y}{dx^2} + 2x^2\frac{dy}{dx} + 4x\frac{dy}{dx} + 2x^2y + 4xy + 2y = e^{-x}$

$\left(x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y\right) + 2\left(x^2\frac{dy}{dx} + 2xy\right) + 2x^2\left(\frac{z}{x^2}\right) = e^{-x}$

Subst $\quad \frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x}$ — (2)

(8) contd CF: $m^2+2m+2=0$

$$(m+1)^2+1=0$$

$$m=-1\pm i$$

$\therefore$ CF ~~of~~ $z=e^{-x}(A\cos x+B\sin x)$

PI try ~~of~~ $z=\lambda e^{-x}$ $\quad \frac{dz}{dx}=-\lambda e^{-x}$ $\quad \frac{d^2z}{dx^2}=\lambda e^{-x}$

in (2) $\lambda e^{-x}-2\lambda e^{-x}+2\lambda e^{-x}=e^{-x}$

$$\lambda=1$$

$\therefore$ General Solution $z=e^{-x}(A\cos x+B\sin x)+e^{-x}$

but ~~of~~ $z=yx^2$

$$y=\frac{e^{-x}}{x^2}(A\cos x+B\sin x+1)$$

(9) $\cos x\frac{d^2y}{dx^2}+\sin x\frac{dy}{dx}-2y\cos^3 x=2\cos^5 x$ — (1)

$z=\sin x$ $\qquad$ I need $\frac{dy}{dz}$ and $\frac{d^2y}{dz^2}$

$$\frac{dy}{dz}=\frac{dy}{\bigcirc}\cdot\frac{\bigcirc}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}$$

~~$\frac{dx}{dz}=$~~ $\frac{dz}{dx}=\cos x$ $\quad\therefore\quad \frac{dx}{dz}=\frac{1}{\cos x}$

$\therefore$ $\frac{dy}{dz}=\frac{1}{\cos x}\cdot\frac{dy}{dx}$ $\Longrightarrow$ $\frac{dy}{dx}=\cos x\frac{dy}{dz}$

Now $\frac{d^2y}{dz^2}=\frac{d}{dz}\left(\frac{dy}{dz}\right)=\frac{d}{dz}\left(\frac{1}{\cos x}\frac{dy}{dx}\right)=\frac{d}{dz}\left(\underbrace{\sec x}_{p}\,\underbrace{\frac{dy}{dx}}_{q}\right)$

Q9 Alternative

$$\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x \quad — (1)$$

$$z = \sin x \implies \frac{dz}{dx} = \cos x$$

$$\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{dy}{dx}\cdot\frac{1}{\cos x} \implies \frac{dy}{dx} = \cos x \frac{dy}{dz}$$

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\underset{p}{\sec x}\cdot\underset{q}{\frac{dy}{dx}}\right)$$

$$\frac{d^2y}{dz^2} = p\frac{dq}{dz} + q\frac{dp}{dz}$$

$$= \sec x\cdot\left(\frac{dq}{\bigcirc}\cdot\frac{\bigcirc}{dz}\right) + \frac{dy}{dx}\left(\frac{dp}{\bigcirc}\cdot\frac{\bigcirc}{dz}\right)$$

$$= \sec x\left(\frac{dq}{dx}\cdot\frac{dx}{dz}\right) + \frac{dy}{dx}\left(\frac{dp}{dx}\cdot\frac{dx}{dz}\right)$$

$$= \sec x\left(\frac{d^2y}{dx^2}\cdot\frac{1}{\cos x}\right) + \frac{dy}{dx}\left(\sec x\tan x\cdot\frac{1}{\cos x}\right)$$

$$\frac{d^2y}{dz^2} = \sec^2 x\frac{d^2y}{dx^2} + \sec^2 x\tan x\frac{dy}{dx}$$

÷ (1) by $\cos^4 x$

$$\sec^2 x\frac{d^2y}{dx^2} + \sec^2 x\tan x\frac{dy}{dx} - 2y = 2\cos^2 x$$

$$\frac{d^2y}{dz^2} - 2y = 2\cos^2 x$$

but $\sin^2 x = z^2$

$1 - \cos^2 x = z^2$

$\cos^2 x = 1 - z^2$

$$\therefore \frac{d^2y}{dz^2} - 2y = 2(1-z^2) \text{ As required}$$

(9) So $\frac{dy}{dx} = \cos x \frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\underbrace{\cos x}_{p} \underbrace{\frac{dy}{dz}}_{q}\right)$$

$$\frac{d^2y}{dx^2} = \cos x \frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz} \cdot -\sin x$$

$$\frac{d^2y}{dx^2} = \cos x\left(\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}\right) - \sin x \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \cos x\left(\frac{d^2y}{dz^2} \cdot \cos x\right) - \sin x \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}$$

Implicit Diff using chain rule...

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$$

$$= 2y \cdot \frac{dy}{dx}$$

So $\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$$

$$= \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

Sub in ① $\cos x\left(\cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}\right) + \sin x\left(\cos x \frac{dy}{dz}\right) - 2y\cos^3 x = 2\cos^5 x$

$$\cos^3 x \frac{d^2y}{dz^2} - \cos x \sin x \frac{dy}{dz} + \cos x \sin x \frac{dy}{dz} - 2y\cos^3 x = 2\cos^5 x$$

$\div \cos^3 x$ $\quad \frac{d^2y}{dz^2} - 2y = 2\cos^2 x = 2(1-\sin^2 x) = 2(1-z^2) = 2-2z^2$ ②

CF: ~~$m^2 - 2m = 0$, $m(m-2) = 0$, $m=0$, $m=2$, $\therefore$ CF $y = A + Be^{2z}$~~

$m^2 - 2 = 0$

$m = \pm\sqrt{2}$

$\therefore$ CF $y = Ae^{z\sqrt{2}} + Be^{-z\sqrt{2}}$

PI: Try $y = \lambda + \mu z + \nu z^2$

$$\frac{dy}{dz} = \mu + 2\nu z$$

$$\frac{d^2y}{dz^2} = 2\nu$$

(9) Cont'd in (2) $2\nu - 2(\lambda + \mu z + \nu z^2) = 2 - 2z^2$

Comp coeff's z^2: $-2\nu = -2$

$\nu = 1$

z: $-2\mu = 0$

$\mu = 0$

: $2\nu - 2\lambda = 2$

$\nu - \lambda = 1$

$1 - \lambda = 1$

$\lambda = 0$

$\therefore$ PI $y = z^2$

$\therefore$ General Solution $y = Ae^{\cancel{z}\sqrt{2}} + Be^{-z\sqrt{2}} + z^2$

but $z = \sin x$

$\therefore$ $y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x.$