

Unit 3 – Proof

In mathematics it is tempting on the basis of checking a number of special cases to deduce that a general conjecture is true.

Consider the function $f(n) = n^3 - 4n^2 + 7n + 1$, where n is a positive integer.

$$f(1) = 5$$

$$f(2) = 7$$

$$f(3) = 13$$

$$f(4) = 29$$

$$f(5) = 61$$

all of these numbers are primes.

A possible conjecture would therefore be that *when n is an integer, $n^3 - 4n^2 + 7n + 1$ is a prime*.

However, although this conjecture is true for $n = 1, 2, 3, 4, 5$ it may not be true for all integer values.

Indeed $f(6) = 115$, which is not prime.

A correct proof is the only way to convince another of the truth of a conjecture. There are a number of methods of proof.

In A level, you will have met examples of direct proof such as the sums of arithmetic and geometric series. Other methods of proof include mathematical induction and disproof by counter-example (which is what $f(6)$ achieves above). In Unit 3, we focus on proof by contradiction.

Proof by Contradiction

Proof is concerned with the demonstration of the truth of an assertion. The essence of proof by contradiction is to assume the assertion is false and show that the assumption leads to a contradiction.

Eg1 Prove that if n^2 is even, then n is even

Eg2 Prove that $\sqrt{2}$ is irrational

Eg3 Use a proof by contradiction to show that if a and b are real numbers, then $a^2 + b^2 \geq 2ab$

Eg4 Prove by contradiction that if x is real and $x > 0$ then $x + \frac{4}{x} \geq 4$

Eg5 Prove that there is an infinite amount of prime numbers

Eg1 If n^2 is even, then n is even.

assume that n is not even, ie n is odd

If n is odd then $n = 2k+1$, For k is an integer, n always odd

$$\text{So } n^2 = (2k+1)^2$$

$$n^2 = 4k^2 + 4k + 1$$

\uparrow \uparrow
always even + always even + 1 = always odd

but we are given that n^2 is even, so contradiction

The assumption (n is not even) is FALSE, n is even!

Eg2 If $\sqrt{2}$ is irrational it cannot be expressed as an irreducible fraction $\frac{m}{n}$ where m and n have no common factors.

Assume that $\sqrt{2} = \frac{m}{n}$, ie $\sqrt{2}$ is rational

$$2 = \frac{m^2}{n^2}$$

$$2n^2 = m^2 \quad - \textcircled{1}$$

$2n^2$ is a multiple of 2 \therefore even, so m^2 is even

If m^2 is even, so is m

As m is even $m = 2k$ where k is an integer

$$\text{In } \textcircled{1} \quad 2n^2 = (2k)^2 = 4k^2$$

So $n^2 = 2k^2$, as before $2k^2$ even, $\frac{n^2}{2}$ even, $\frac{k^2}{2}$ even

So m is even & n is even, so they have a common factor which is a contradiction.

\therefore Assumption is FALSE, $\sqrt{2}$ is irrational.

Eg3 Show that $a^2 + b^2 \geq 2ab$ for $a, b \in \mathbb{R}$

Assume that $a^2 + b^2 < 2ab$

$$\text{then } a^2 + b^2 - 2ab < 0$$

$$(a-b)^2 < 0$$

If a, b are real then $a-b$ is real & $(a-b)^2$ cannot be -ve

\therefore contradiction

\therefore assumption is FALSE and $a^2 + b^2 \geq 2ab$

Eg4 $x + \frac{4}{x} \geq 4$ for $x \in \mathbb{R} \text{ & } x > 0$

Assume that $x + \frac{4}{x} < 4$

$$x^2 + 4 < 4x$$

$$x^2 - 4x + 4 < 0$$

$$(x-2)^2 < 0$$

$(x-2)^2$ cannot be -ve for all $x \in \mathbb{R}$ \therefore contradiction

Assumption is False and $x + \frac{4}{x} \geq 4$

Eg) Infinite amount of prime n°)

Assume that there are a finite amount

$p_1, p_2, p_3, p_4, \dots, p_n$

numbers are generated by multiples of primes

eg $2 \times 3 \times 7 = 42$

define a new number as product of all primes + 1.

i.e $M = p_1 \times p_2 \times p_3 \times \dots \times p_n + 1$

∴ M is larger than any of the primes.

so can't be prime, so must be divisible by at least one
of the finite list of primes

but $M \div$ prime will always have a remainder of 1 {that's the +1}

So must be prime - CONTRADICTION

Assumption is FALSE, there are an infinite n° of primes.

Exercise (WJEC PPQs)

1. Prove by contradiction the following proposition.

If a, b are positive real numbers, then $a + b \geqslant 2\sqrt{ab}$.

The first line of the proof is given below.

Assume that positive real numbers a, b exist such that $a + b < 2\sqrt{ab}$. [3]

2. Prove by contradiction the following proposition.

When x is real and positive,

$$4x + \frac{9}{x} \geqslant 12.$$

The first line of the proof is given below.

Assume that there is a positive and real value of x such that

$$4x + \frac{9}{x} < 12. \quad [3]$$

3. Complete the following proof by contradiction to show that $\sqrt{5}$ is irrational.

Assume that $\sqrt{5}$ is rational. Then $\sqrt{5}$ may be written in the form $\frac{a}{b}$, where a, b are integers having no common factors.

$$\therefore a^2 = 5b^2.$$

$\therefore a^2$ has a factor 5.

$\therefore a$ has a factor 5 so that $a = 5k$, where k is an integer.

[3]

4. Prove by contradiction the following proposition.

When x is real,

$$(5x - 3)^2 + 1 \geqslant (3x - 1)^2.$$

The first line of the proof is given below.

Assume that there is a real value of x such that

$$(5x - 3)^2 + 1 < (3x - 1)^2. \quad [3]$$

Exercise

$$\textcircled{1} \quad a+b \geq 2\sqrt{ab}$$

Assume that positive real numbers a, b exist such that $a+b < 2\sqrt{ab}$

$$(a+b)^2 < 4ab$$

$$a^2 + b^2 + 2ab < 4ab$$

$$a^2 + b^2 - 2ab < 0$$

$$(a-b)^2 < 0$$

for real numbers $(a-b)^2$ cannot be < 0 \therefore contradiction

assumption is FALSE, $a+b \geq 2\sqrt{ab}$

$$\textcircled{2} \quad 4x + \frac{9}{x} \geq 12 \quad x > 0, x \in \mathbb{R}$$

$$\text{Assume } 4x + \frac{9}{x} < 12$$

$$4x^2 + 9 < 12x$$

$$4x^2 - 12x + 9 < 0$$

$$(2x-3)^2 < 0$$

Contradicts $x \in \mathbb{R}$

assumption is false, $4x + \frac{9}{x} \geq 12$

(3) Assume that $\sqrt{5}$ is rational. Then $\sqrt{5}$ may be written in the form $\frac{a}{b}$, where a, b are integers having no common factors.

$$\therefore a^2 = 5b^2 \quad \text{--- (1)}$$

$\therefore a^2$ has a factor of 5

$\therefore a$ has a factor of 5 so that $a = 5k$, where a is an integer

Sub in (1) $(5k)^2 = 5b^2$

$$25k^2 = 5b^2$$

So b^2 has a factor of 5

b has a factor of 5

$\therefore a$ & b both have factors of 5 — CONTRADICTION

assumption is FALSE, $\sqrt{5}$ is irrational.

$$(4) \quad (5x-3)^2 + 1 \geq (3x-1)^2$$

Assume that there is a real value of x such that

$$(5x-3)^2 + 1 < (3x-1)^2$$

$$25x^2 - 30x + 9 + 1 < 9x^2 - 6x + 1$$

$$16x^2 - 24x + 9 < 0$$

$$(4x-3)^2 < 0$$

↑ contradicts $x \in \mathbb{R}$

∴ Assumption is false hence $(5x-3)^2 + 1 \geq (3x-1)^2$